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Signed bicyclic graphs minimizing the least Laplacian eigenvalue

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ABSTRACT

A signed graph is a pair $\Gamma = (G, \sigma)$, where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is the sign function on the edges of G. For a signed graph we consider the Laplacian matrix defined as $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the matrix of vertices degrees of G and $A(\Gamma)$ is the (signed) adjacency matrix. The least Laplacian eigenvalue is zero if and only if the signed graph is balanced, i.e. all cycles contain an even number of negative edges. Here we show that among the unbalanced bicyclic signed graphs of given order $n \geq 5$ the least Laplacian eigenvalue is minimal for signed graphs consisting of two triangles, only one of which is unbalanced, connected by a path. We also identify the signed graphs minimizing the least eigenvalue among those whose unbalanced (bicyclic) base is a theta-graph.

1. Introduction

A *signed graph* Γ is a pair (G, σ), where $G = (V(G), E(G))$ is a simple graph and $\sigma : E(G) \to \{+1, -1\}$ is a sign function (or *signature*) on the edges of G. The (unsigned)

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graph G of $\Gamma = (G, \sigma)$ is called the *underlying graph*. Let C be a cycle in Γ, the *sign* of C is given by $\mathrm{sign}(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (resp. −1) is called *positive* (resp. *negative*); alternatively, we can say that a cycle is positive if it contains an even number of negative edges. A signed graph is *balanced* if all cycles are positive; otherwise it is *unbalanced* [9]. If all edges in Γ are positive (negative), then Γ is denoted by $(G, +)$ (resp. $(G, -)$), in this case we refer to such signature as the all-positive (resp. all-negative) one. Signed graphs might be seen as weighted graphs with edge weights equal to ±1, however the theory of signed graphs and that one of weighted graphs do not completely overlap in view of the cycle sign. In particular, signed graphs can be regarded as special kind of gain graphs and biased graphs [12,13].

Most of the concepts defined for (unsigned) graphs are directly extended to signed graphs. For example, the degree of a vertex v in G, denoted by $\deg(v)$, is also its degree in Γ. The order of Γ is the order of G and it is denoted by $|\Gamma|$. A signed graph is k-cyclic if the underlying graph is k-cyclic, which means that it is connected and $|E(G)| = |V(G)| + k - 1$. Furthermore, if some subgraph of the underlying graph is under consideration, then the sign function for the subgraph is the restriction of the original one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (note, all edges incident to v are deleted). If $U \subset V(G)$ then $\Gamma[U]$ (with underlying graph $G[U]$) denotes the (signed) induced subgraph arising from U, whereas $\Gamma - U = \Gamma[V(G) \setminus U]$. We also write $\Gamma - \Gamma[U]$ instead of $\Gamma - U$.

For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) switching equivalent to Γ, and it is denoted by $\Gamma^U \sim \Gamma$. In this case we write $\sigma_{\Gamma^U} \sim \sigma_\Gamma$. In fact, switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent. Observe that switchings do not change cycle signs, therefore switching equivalent graphs share the same set of positive cycles.

Signed graphs, as the unsigned ones, can be studied by means of matrix theory. For a signed graph Γ, we consider the *Laplacian matrix* $L(= L(\Gamma)) = D - A$, where D is the diagonal matrix $\mathrm{diag}(d_1, d_2, \dots, d_n)$ of vertex degrees, $A = (a_{ij})$ is the (signed) *adjacency matrix* (i.e. $a_{ij} = \sigma(ij)$ if vertices i and j are adjacent, and 0 otherwise). The Laplacian matrix is a real, symmetric and semidefinite positive matrix (see also Section 2). The *Laplacian polynomial* is $\det(xI - L) = \phi(\Gamma, x)$ and the roots of $\phi(\Gamma, x)$ are also called the *Laplacian eigenvalues* (or L-eigenvalues) of Γ. Since L is real and symmetric, the L-eigenvalues are real; moreover their algebraic and geometric multiplicities coincide. Together with their multiplicities, they comprise the *Laplacian spectrum* of Γ. A non-zero vector $\mathbf{x}$ satisfying the equation $L\mathbf{x} = \mu\mathbf{x}$ is an eigenvector (or μ-eigenvector) of L, and also of Γ if it is considered as a vertex-labeled signed graph. The eigenspace of L for μ is the set $\mathcal{E}(\mu; \Gamma) = \{\mathbf{x} : L\mathbf{x} = \mu\mathbf{x}\}$; it is also an eigenspace of Γ for μ with respect to L.