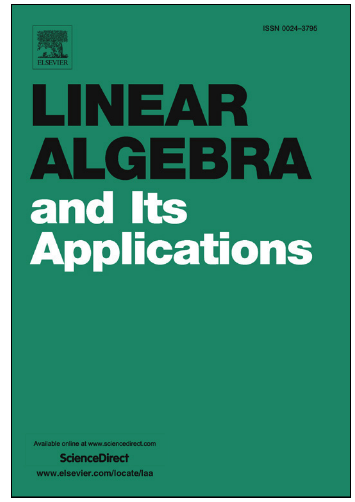

A contraction theorem for the largest eigenvalue of a multigraph

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PII: S0024-3795(18)30366-5
DOI: https://doi.org/10.1016/j.laa.2018.07.032
Reference: LAA 14674

To appear in: *Linear Algebra and its Applications*

Received date: 9 March 2018
Accepted date: 26 July 2018

Please cite this article in press as: J. McKee, A contraction theorem for the largest eigenvalue of a multigraph, *Linear Algebra Appl.* (2018), https://doi.org/10.1016/j.laa.2018.07.032

A CONTRACTION THEOREM FOR THE LARGEST EIGENVALUE OF A MULTIGRAPH

JAMES MCKEE

ABSTRACT. Let G be a multigraph with loops, and let e be an edge in G. Let H be the multigraph obtained by contracting along the edge e. Let λ_G and λ_H be the largest eigenvalues of G and H respectively. A characterisation theorem is given of precisely when $\lambda_H < \lambda_G$, $\lambda_H = \lambda_G$, or $\lambda_H > \lambda_G$. In the case where H happens to be a simple graph, then so is G, and the theorem subsumes those of Hoffman-Smith and Gumbrell for subdivision of edges or splitting of vertices of a graph.

1. INTRODUCTION

Let H be a graph, and let e be an edge on an internal path (definitions will come later). Hoffman and Smith [3] showed that if one subdivides e to produce a graph G with an extra vertex, splitting e into two edges, then the largest eigenvalue goes down, unless the largest eigenvalue of H was equal to 2, in which case the largest eigenvalue of G is also 2. By contrast, if one subdivides an edge that is not on an internal path, then the largest eigenvalue goes up. Gumbrell extended this to the splitting of a vertex not on an internal path [2].

Reversing the subdivision process in either case, one moves from G to H by contracting along an edge. If one considers contracting along an arbitrary edge of a graph G, one might produce a multigraph H rather than a simple graph: multiple edges may appear. If one considers contracting along an edge of a multigraph, one might meet loops. Thus an attempt to unify and extend the work of Hoffman-Smith and Gumbrell by reversing their subdivision process naturally leads to working with multigraphs with loops. In this setting, we prove a general contraction theorem, Theorem 5, which describes precisely when the largest eigenvalue increases, decreases, or stays the same when an edge is contracted. This theorem subsumes the theorems of Hoffman-Smith and Gumbrell, and generalises them. Even in the setting of graphs it covers some cases not included in their theorems (although these extra graph cases are all trivial consequences of Perron-Frobenius theory).

The plan of the paper is as follows. First we make precise what we mean by a multigraph, define the operations of coalescing vertices and contracting along an edge, and recall some background Perron-Frobenius theory. Then we extend Smith's classification [6] of connected graphs that have largest eigenvalue at most 2 to the realm of multigraphs. We then state the main contraction theorem, Theorem 5, and derive some easy consequences: the theorems of Hoffman-Smith and Gumbrell, and (after proving a general coalescing theorem, Theorem 6) Simić's vertex-splitting theorem [5]. The final section of the paper gives the proof of the main theorem.

1991 *Mathematics Subject Classification.* 05C50.
Key words and phrases. graph spectra, Hoffman and Smith, edge contraction.