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Characterization of rational matrices that admit finite digit representations ☆

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ABSTRACT

Let A be an $n \times n$ matrix with rational entries and let $\mathbb{Z}^n[A] := \bigcup_{k=1}^{\infty} \left(\mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^{k-1}\mathbb{Z}^n\right)$ be the minimal A-invariant $\mathbb{Z}$-module containing the lattice $\mathbb{Z}^n$. If $\mathcal{D} \subset \mathbb{Z}^n[A]$ is a finite set we call the pair $(A, \mathcal{D})$ *a digit system*. We say that $(A, \mathcal{D})$ has *the finiteness property* if each $\mathbf{z} \in \mathbb{Z}^n[A]$ can be written in the form $\mathbf{z} = \mathbf{d}_0 + A\mathbf{d}_1 + \cdots + A^k\mathbf{d}_k$, with $k \in \mathbb{N}$ and *digits* $\mathbf{d}_j \in \mathcal{D}$ for $0 \leq j \leq k$. We prove that for a given matrix $A \in M_n(\mathbb{Q})$ there is a finite set $\mathcal{D} \subset \mathbb{Z}^n[A]$ such that $(A, \mathcal{D})$ has the finiteness property if and only if A has no eigenvalue of absolute value < 1. This result is the matrix analogue of *the height reducing property* of algebraic numbers. In proving this result we also characterize integer polynomials $P \in \mathbb{Z}[x]$ that admit digit systems having the finiteness property in the quotient ring $\mathbb{Z}[x]/(P)$.

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Let A be an $n \times n$ integer matrix and let $\mathcal{D} \subset \mathbb{Z}^n$ be finite. The pair $(A, \mathcal{D})$ is called *a digit system* in the lattice $\mathbb{Z}^n$. The matrix A is called *the base* of the digit system, the set $\mathcal{D}$ is called its *digit set*. The pair $(A, \mathcal{D})$ is said to have *the finiteness property*, if every vector $\mathbf{z} \in \mathbb{Z}^n$ can be written as a finite sum using only the digits from $\mathcal{D}$, multiplied by non-negative powers of A, *i.e.*, if every vector $\mathbf{z} \in \mathbb{Z}^n$ admits *a radix representation*

$$\mathbf{z} = \mathbf{d}_0 + A\mathbf{d}_1 + \cdots + A^k\mathbf{d}_k, \tag{1}$$

with *digits* $\mathbf{d}_j \in \mathcal{D}$ for $0 \le j \le k$. Such representations, in general, need not be unique. If they are, we say that $(A, \mathcal{D})$ has *the unique representation property*. If one defines the set $\mathcal{D}[A] \subset \mathbb{R}^n$ by

$$\mathcal{D}[A] := \{\mathbf{d}_0 + A\mathbf{d}_1 + \cdots + A^k\mathbf{d}_k \in \mathbb{R}^n, \mathbf{d}_j \in \mathcal{D}, 0 \le j \le k,\ j, k \in \mathbb{Z}\},$$

then the finiteness property of the pair $(A, \mathcal{D})$ can be restated as $\mathbb{Z}^n = \mathcal{D}[A]$.

For a ring $R \subset \mathbb{R}$ in all what follows we will denote $M_n(R)$ the set of $n \times n$ matrices with entries taken from R. Recall that a matrix of $M_n(R)$ is called *expanding*, if each of its eigenvalues is strictly greater than 1 in absolute value. In 1993 Vince [26,27] demonstrated that for each expanding integer matrix $A \in M_n(\mathbb{Z})$, there exists a finite digit set $\mathcal{D}$, consisting of integer vectors, such that $(A, \mathcal{D})$ is a digit system in $\mathbb{Z}^n$ with finiteness property (this result is essentially contained in [27, Lemma 2], although it was not stated in his paper explicitly in this form). The basic underlying principle behind the finiteness property is the ultimate periodicity of the mapping $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$, $\Phi(\mathbf{x}) = A^{-1}(\mathbf{x} - \mathbf{d}(\mathbf{x}))$, where $\mathbf{d}(\mathbf{x}) \in \mathcal{D}$ satisfies $\mathbf{x} \equiv \mathbf{d} \pmod{A\mathbb{Z}^n}$. Vince also noted that, when A has at least one eigenvalue of absolute value < 1, then $(A, \mathcal{D})$ cannot have the finiteness property. Moreover, in [27, Proposition 4] he showed that a digit system cannot possess the unique representation property unless A is expanding. We refer to A. Kovács [15], where problems on this topic are formulated. Although we attribute the matrix version formulation to Vince, all basic principles were understood much earlier in the context of number systems defined in orders of number fields. With special emphasis on the unique representation property, these number systems were studied extensively by Kátai and Szabó [14], Kátai and B. Kovács [12,13,18], Gilbert [10,11], B. Kovács and Pethő [19,20], Burcsi and A. Kovács [8], Akiyama and Rao [2], Scheicher [23], and many others. More recently, the set of algebraic numbers α that admit number systems in $\mathbb{Z}[\alpha]$ with finiteness property was investigated and fully characterized in the series of papers [1,3–5] by Akiyama and his co-authors. In this context the finiteness property is also known as *the height reducing property* of the minimal polynomial of α. We mention that the characterization of the unique representation property is far from being complete.

In the present note, we extend Vince's results on the finiteness property in two directions: firstly, we deal with cases when A has rational (not necessarily integer) entries; secondly, we deal with the situation when A has eigenvalues $|\lambda| = 1$.