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On the algebraic boundaries among typical ranks for real binary forms

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A B S T R A C T

We describe the algebraic boundaries of the regions of real binary forms with fixed typical rank and of degree at most eight, showing that they are dual varieties of suitable coincident root loci.

1. Introduction

The study of symmetric tensors, of their rank, decomposition and identifiability is a classical problem, which received great attention recently in both pure and applied mathematics; see e.g. [18] and references therein, see also [2,3,21,8,20,1].

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Symmetric tensors can be interpreted as homogeneous polynomials, also called forms. The rank of a degree d form f is the minimum integer r such that there exists a decomposition $f = \sum_{i=1}^{r} c_i(l_i)^d$, where l_i are linear forms and c_i are scalars.

In this paper we focus on the case of binary forms over the field of real numbers $\mathbb{R}$. In this case it is known that the (real) rank of a general form satisfies the inequalities $\frac{d+1}{2} \leq r \leq d$. Moreover all the ranks in this range are typical, that is, they occur in open subsets (with respect to the Euclidean topology) of the real vector space of degree d forms; see [4].

A natural problem is to understand the geometry of the sets $\Omega_{d,r}$ of forms of degree d and rank r. In particular we would like to describe the boundaries among the various sets of forms of given typical rank; more precisely, we are interested in understanding the algebraic boundaries, i.e., the Zariski closures of the topological boundaries (see Section 3 for the precise definitions).

The easiest case is the maximal one, that is when the rank is equal to the degree. Indeed it is proved in [10,9] that a binary form of degree d with distinct roots has rank d if and only if all its roots are real. Hence its algebraic boundary is the discriminant hypersurface of forms with two coincident roots.

The geometric description of the sets $\Omega_{d,r}$ becomes much more intricate for $r < d$. Indeed, although the rank of a form is always greater than or equal to the number of its real distinct roots, in general the number of real distinct roots is not invariant in the region $\Omega_{d,r}$.

In [19] the authors study the boundary of the set of forms of rank $\lceil \frac{d+1}{2} \rceil$, which is the minimal typical rank. They prove that the components of the boundary are dual varieties of suitable coincident root loci.

We tackle the problem of describing all the intermediate boundaries in general, as proposed by Lee and Sturmfels in [19, Remark 4.5]. Our approach provides a unified description of all the boundaries in terms of dual varieties of coincident root loci. We recall that the cases of degree $d \leq 5$ have been described in [9], while the case $d = 6$ follows by [19,10] (see Proposition 3.1 for more details). In this paper we focus on the cases $d = 7$ and $d = 8$, and we postpone a general description to future work.

The paper is organized as follows. Sections 2 and 3 are devoted to preliminary results; in particular, in Proposition 3.1 we recall the known results concerning algebraic boundaries for real binary forms of degree less than or equal to 6. Section 4 and 5 contain our main results, which are Theorem 4.1 and Theorem 5.1, describing the algebraic boundaries for real binary forms of degrees respectively 7 and 8. They turn out to be dual varieties of suitable coincident root loci. Finally in Section 6 we explain some of the computational methods of which we take advantage in our study.

2. Coincident root loci

We recall here some known results on coincident root loci, referring to [26,15,6,7,17] for details.