

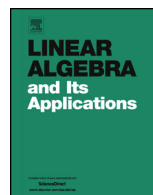


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# Chip-firing groups of iterated cones



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## ABSTRACT

Let  $\Gamma$  be a finite graph and let  $\Gamma_n$  be the “ $n$ th cone over  $\Gamma$ ” (i.e., the join of  $\Gamma$  and the complete graph  $K_n$ ). We study the asymptotic structure of the chip-firing group  $\text{Pic}^0(\Gamma_n)$ .

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## 1. Introduction

The *chip-firing* groups  $\text{Pic}^0(\Gamma) \subset \text{Pic}(\Gamma)$  of a finite graph  $\Gamma$  are classical objects of combinatorial study. Baker [2] developed the connection between line bundles on a semistable arithmetic curve  $\mathcal{X}$  and  $\text{Pic}^0(\Gamma)$ , where  $\Gamma$  is the dual graph of the special fiber of  $\mathcal{X}$ , and with various coauthors [6,7] discovered that the cornerstone theorems satis-

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fied by algebraic curves (e.g., Riemann–Roch and Clifford’s theorem) admit non-trivial analogous theorems for graphs.

The technology transfer flows both ways; chip-firing (and variants and tools from tropical geometry) have emerged as a central tool in recent results across several sub-fields of algebraic/arithmetic geometry and number theory, including the maximal rank conjecture for quadrics [13], the Gieseker–Petri theorem [12], the Brill–Noether theorem [10], and the uniform boundedness conjecture [14]; see [3] for an extensive survey.

An interest in the *computational* properties of  $\text{Pic}^0(\Gamma)$  has recently emerged. Several authors, including [5,11,15], have worked to compute  $\text{Pic}^0(\Gamma)$  (or, failing that,  $|\text{Pic}^0(\Gamma)|$ , which is equal to the number of spanning trees of  $\Gamma$  [8, Theorem 6.2]) for various families of graphs; we refer the reader to [1, pg. 1155] for nearly a complete list of authors contributing to this area.

Our question of interest is the behavior of the chip-firing group of the  $n$ th cone  $\Gamma_n$  over  $\Gamma$ , where  $\Gamma_n$  is defined as the join of  $\Gamma$  with the complete graph  $K_n$ . Recall, the join of two graphs  $\Gamma_1$  and  $\Gamma_2$  is a graph obtained from  $\Gamma_1$  and  $\Gamma_2$  by joining each vertex of  $\Gamma_1$  to all vertices of  $\Gamma_2$ . In [1], the authors interpret the chip-firing group of the  $n$ th cone of the Cartesian product of graphs as a function of the chip-firing group of the cone of their factors. As a consequence, they completely describe the chip-firing group of the  $n$ th cone over the  $d$ -dimensional hypercube.

Our main theorem concerns the chip-firing group of the  $n$ th cone over a fixed graph.

**Theorem A.** *Let  $\Gamma$  be a graph on  $k \geq 1$  vertices. Let  $n \geq 1$  be an integer, and let  $\Gamma_n$  be the  $n$ th cone over  $\Gamma$  defined above. Then there is a short exact sequence of abelian groups*

$$0 \rightarrow (\mathbb{Z}/(n+k)\mathbb{Z})^{n-1} \rightarrow \text{Pic}^0(\Gamma_n) \rightarrow H_n \rightarrow 0$$

where the order of  $H_n$  is  $|P_\Gamma(-n)|$  and  $P_\Gamma(x)$  is the characteristic polynomial of the rational Laplacian operator.

In particular, this immediately gives an exact formula for the number of spanning trees of  $\Gamma_n$ .

**Corollary B.** *Let  $\Gamma$  be a graph on  $k \geq 1$  vertices. Let  $n \geq 1$  be an integer, and let  $\Gamma_n$  be the  $n$ th cone over  $\Gamma$  defined above. There is a subgroup of  $\text{Pic}^0(\Gamma_n)$  isomorphic to  $(\mathbb{Z}/(n+k)\mathbb{Z})^{n-1}$ , and*

$$|\text{Pic}^0(\Gamma_n)| = (n+k)^{n-1}|P_\Gamma(-n)|$$

where  $P_\Gamma(x)$  is the characteristic polynomial of the rational Laplacian operator.

**Remark 1.1.** In a previous version of this paper, the authors erroneously claimed that this exact sequence was split for odd values of  $n+k$ , and conjectured it was split in general. We are very grateful to Gopal Goel for pointing out this error and providing a counter

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