Linear Algebra and its Applications 556 (2018) 46–54

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Chip-firing groups of iterated cones

Morgan V. Brown [a], Jackson S. Morrow [b,*], David Zureick-Brown [b]

[a] Department of Mathematics, University of Miami, Coral Gables, FL 33146 USA
[b] Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

ARTICLE INFO

Article history:
Received 14 September 2017
Accepted 21 June 2018
Available online 25 June 2018
Submitted by R. Brualdi

MSC:
primary 05C50
secondary 05C25

Keywords:
Graph Laplacian
Critical group
Abelian sandpile
Chip-firing
Matrix-tree theorem

ABSTRACT

Let Γ be a finite graph and let Γ_n be the "nth cone over Γ" (i.e., the join of Γ and the complete graph K_n). We study the asymptotic structure of the chip-firing group $\mathrm{Pic}^0(\Gamma_n)$.

Published by Elsevier Inc.

1. Introduction

The *chip-firing* groups $\mathrm{Pic}^0(\Gamma) \subset \mathrm{Pic}(\Gamma)$ of a finite graph Γ are classical objects of combinatorial study. Baker [2] developed the connection between line bundles on a semistable arithmetic curve $\mathscr{X}$ and $\mathrm{Pic}^0(\Gamma)$, where Γ is the dual graph of the special fiber of $\mathscr{X}$, and with various coauthors [6,7] discovered that the cornerstone theorems satis-

* Corresponding author.
E-mail addresses: mvbrown@math.miami.edu (M.V. Brown), jmorrow4692@gmail.com (J.S. Morrow), dzb@mathcs.emory.edu (D. Zureick-Brown).

https://doi.org/10.1016/j.laa.2018.06.023
0024-3795/Published by Elsevier Inc.

fied by algebraic curves (e.g., Riemann–Roch and Clifford's theorem) admit non-trivial analogous theorems for graphs.

The technology transfer flows both ways; chip-firing (and variants and tools from tropical geometry) have emerged as a central tool in recent results across several subfields of algebraic/arithmetic geometry and number theory, including the maximal rank conjecture for quadrics [13], the Gieseker–Petri theorem [12], the Brill–Noether theorem [10], and the uniform boundedness conjecture [14]; see [3] for an extensive survey.

An interest in the *computational* properties of $\mathrm{Pic}^0(\Gamma)$ has recently emerged. Several authors, including [5,11,15], have worked to compute $\mathrm{Pic}^0(\Gamma)$ (or, failing that, $|\mathrm{Pic}^0(\Gamma)|$, which is equal to the number of spanning trees of Γ [8, Theorem 6.2]) for various families of graphs; we refer the reader to [1, pg. 1155] for nearly a complete list of authors contributing to this area.

Our question of interest is the behavior of the chip-firing group of the nth cone Γ_n over Γ, where Γ_n is defined as the join of Γ with the complete graph K_n. Recall, the join of two graphs Γ_1 and Γ_2 is a graph obtained from Γ_1 and Γ_2 by joining each vertex of Γ_1 to all vertices of Γ_2. In [1], the authors interpret the chip-firing group of the nth cone of the Cartesian product of graphs as a function of the chip-firing group of the cone of their factors. As a consequence, they completely describe the chip-firing group of the nth cone over the d-dimensional hypercube.

Our main theorem concerns the chip-firing group of the nth cone over a fixed graph.

Theorem A. *Let Γ be a graph on $k \geq 1$ vertices. Let $n \geq 1$ be an integer, and let Γ_n be the nth cone over Γ defined above. Then there is a short exact sequence of abelian groups*

$$0 \to (\mathbb{Z}/(n+k)\mathbb{Z})^{n-1} \to \mathrm{Pic}^0(\Gamma_n) \to H_n \to 0$$

where the order of H_n is $|P_\Gamma(-n)|$ and $P_\Gamma(x)$ is the characteristic polynomial of the rational Laplacian operator.

In particular, this immediately gives an exact formula for the number of spanning trees of Γ_n.

Corollary B. *Let Γ be a graph on $k \geq 1$ vertices. Let $n \geq 1$ be an integer, and let Γ_n be the nth cone over Γ defined above. There is a subgroup of $\mathrm{Pic}^0(\Gamma_n)$ isomorphic to $(\mathbb{Z}/(n+k)\mathbb{Z})^{n-1}$, and*

$$|\mathrm{Pic}^0(\Gamma_n)| = (n+k)^{n-1}|P_\Gamma(-n)|$$

where $P_\Gamma(x)$ is the characteristic polynomial of the rational Laplacian operator.

Remark 1.1. In a previous version of this paper, the authors erroneously claimed that this exact sequence was split for odd values of $n+k$, and conjectured it was split in general. We are very grateful to Gopal Goel for pointing out this error and providing a counter