Linear Algebra and its Applications 556 (2018) 100–107

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Graphs with real algebraic co-rank at most two

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ARTICLE INFO

Article history:
Received 15 February 2018
Accepted 3 July 2018
Available online 5 July 2018
Submitted by R. Brualdi

MSC:
05C25
05C50
05E99
13P15
15A03
68W30

Keywords:
Critical ideals
Forbidden induced subgraph
Minimum rank
Laplacian matrix
Zero forcing number

ABSTRACT

Recently, there have been found relations between the algebraic co-rank and the zero forcing number along with the minimum rank. We continue on this direction by giving a characterization of the graphs with real algebraic co-rank at most 2. This implies that for any graph with minimum rank at most 3, its minimum rank is bounded from above by its real algebraic co-rank. This sheds some light on the conjecture that the real minimum rank is bounded from above by the real algebraic co-rank.

1. Introduction

Given a simple graph G and a set of indeterminates $X_G = \{x_u : u \in V(G)\}$, the *generalized Laplacian matrix* $L(G, X_G)$ of G is the matrix whose uv-entry is given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

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https://doi.org/10.1016/j.laa.2018.07.002

where m_{uv} is the number of the edges between vertices u and v. Moreover, if $\mathcal{R}[X_G]$ is the polynomial ring over a commutative ring $\mathcal{R}$ with unity in the variables X_G, then the *critical ideals* of G are the determinantal ideals given by

$$I_i^{\mathcal{R}}(G, X_G) = \langle \text{minors}_i(L(G, X_G)) \rangle \subseteq \mathcal{R}[X_G] \text{ for all } 1 \leq i \leq n,$$

where n is the number of vertices of G and $\text{minors}_i(L(G, X_G))$ is the set of the determinants of the $i \times i$ submatrices of $L(G, X_G)$.

An ideal is said to be *trivial* if it is equal to $\langle 1 \rangle$ $(= \mathcal{R}[X])$. The *algebraic co-rank* $\gamma_{\mathcal{R}}(G)$ of G is the maximum integer i for which $I_i^{\mathcal{R}}(G, X_G)$ is trivial. For simplicity, we might refer to the *real algebraic co-rank* of the graph G to $\gamma_{\mathbb{R}}(G)$. Note that $I_n^{\mathcal{R}}(G, X_G) = \langle \det L(G, X_G) \rangle$ is always non-trivial, and if d_G denotes the degree vector, then $I_n^{\mathcal{R}}(G, d_G) = \langle 0 \rangle$.

Critical ideals were defined in [13] and some interesting properties were found. For instance, it was proven that if H is an induced subgraph of G, then $I_i^{\mathcal{R}}(H, X_H) \subseteq I_i^{\mathcal{R}}(G, X_G)$ for all $i \leq |V(H)|$. Thus $\gamma_{\mathcal{R}}(H) \leq \gamma_{\mathcal{R}}(G)$. Initially, critical ideals were defined as a generalization of the critical group (*a.k.a.* sandpile group), since an evaluation of $I_i^{\mathbb{Z}}(G, X_G)$ at $X_G = d_G$ gives us the greater common divisor of the i-minors of the Laplacian matrix of the graph, which can be used for computing the invariant factors of the Smith normal form of the Laplacian matrix of the graph. See [6, Section 4] or [13, Section 3.3] for the details on the relation between critical ideals and the critical group. Also in [3,15] can be found an account of the main results on sandpile group, and in [16] can be found a survey on the Smith normal form in combinatorics done by Stanley.

In [2], it was explored its relation with the zero forcing number and the minimum rank. We continue on this direction. For this, we recall these well-known concepts.

The *zero forcing game* is a color-change game where vertices can be blue or white. At the beginning, the player can pick a set of vertices B and color them blue while others remain white. The goal is to color all vertices blue through repeated applications of the *color change rule*: If x is a blue vertex and y is the only white neighbor of x, then y is forced to become blue. An initial set of blue vertices B is called a *zero forcing set* if starting with B one can make all vertices blue. The *zero forcing number* $Z(G)$ is the minimum cardinality of a zero forcing set. In the following, $\text{mz}(G) = |V(G)| - Z(G)$.

For a graph G on n vertices, the family $\mathcal{S}_{\mathcal{R}}(G)$ collects all $n \times n$ symmetric matrices with entries in the ring $\mathcal{R}$, whose i, j-entry $(i \neq j)$ is nonzero whenever i is adjacent to j and zero otherwise. Note that the diagonal entries can be any element in the ring $\mathcal{R}$. The *minimum rank* $\text{mr}_{\mathcal{R}}(G)$ of G is the smallest possible rank among matrices in $\mathcal{S}_{\mathcal{R}}(G)$. Here we follow [12, Definition 1] and define the rank of a matrix over a commutative ring with unity as the largest k such that there is a nonzero $k \times k$ minor that is not a zero divisor. In the case of $\mathcal{R} = \mathbb{Z}$, the rank over $\mathbb{Z}$ is the same as the rank over $\mathbb{R}$.

In [7], it was proved that $\text{mz}(G) \leq \text{mr}_{\mathcal{R}}(G)$ for any field $\mathcal{R}$. And in [2], it was proved that $\text{mz}(G) \leq \gamma_{\mathcal{R}}(G)$ for any commutative ring $\mathcal{R}$ with unity. However, the relation between $\text{mr}_{\mathcal{R}}(G)$ and $\gamma_{\mathcal{R}}(G)$ remains not completely understood.