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Some spectral norm inequalities on Hadamard products of nonnegative matrices ☆

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ABSTRACT

Let A and B be nonnegative square matrices of the same order. Denote by $\|\cdot\|$ and $\rho(\cdot)$ the spectral norm and the spectral radius respectively. We prove the following inequalities:

$$\|A \circ B\| \le \|A \circ A\|^{\frac{1}{2}} \|B \circ B\|^{\frac{1}{2}};$$

$$\|A \circ B\| \le \rho^{\frac{1}{2}}\left(A^T B \circ B^T A\right) \le \rho^{\frac{1}{2}}\left(A^T B \circ A^T B\right) \le \rho(A^T B),$$

where $\circ$ denotes the Hadamard product. This interpolates the inequality

$$\|A \circ B\| \le \rho(A^T B)$$

due to Huang. Some spectral norm inequalities for an arbitrarily finite number of nonnegative square matrices are also obtained, which refine some other results of Huang.

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1. Introduction and preliminaries

Let M_n denote the set of complex matrices of order n. For matrices $A = (a_{ij}), B = (b_{ij}) \in M_n$, we denote by $\rho(A)$ the spectral radius of A, by $A \circ B = (a_{ij}b_{ij})$ the Hadamard product of A and B. The notation $A \leq B$ means that $B - A$ is entrywise nonnegative. For $T \in M_n$, the *singular values* of T, denoted by $s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T)$, are the eigenvalues of the positive semidefinite matrix $|T| = (T^*T)^{\frac{1}{2}}$. It follows that the singular values of a normal matrix are just the moduli of its eigenvalues. Denote by $\|A\|$ the spectral norm of $A \in M_n$, which equals the largest singular value.

Zhan [11] conjectured that $\rho(A \circ B) \leq \rho(AB)$ for nonnegative matrices $A, B \in M_n$. This conjecture was confirmed by Audenaert in [1] by proving

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}((A \circ A)(B \circ B)) \leq \rho(AB). \tag{1.1}$$

Using the fact that the Hadamard product of two matrices is a principal submatrix of the Kronecker product, Horn and Zhang proved [4] the inequalities

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}(AB \circ BA) \leq \rho(AB). \tag{1.2}$$

Huang [5] generalized the inequality $\rho(A \circ B) \leq \rho(AB)$ to an arbitrarily finite number of nonnegative matrices:

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leq \rho(A_1 A_2 \cdots A_k). \tag{1.3}$$

Over the years, various generalizations on the spectral radius of Hadamard products of nonnegative matrices have been considered in the literature; e.g., [2,6–10].

The aim of this paper is to present some inequalities on the spectral norm of the Hadamard product of nonnegative matrices. The main results are the following

(1) Let $A, B \in M_n$ be nonnegative matrices. Then

$$\|A \circ B\| \leq \|A \circ A\|^{\frac{1}{2}} \|B \circ B\|^{\frac{1}{2}};$$

Denote by S_k the set of all permutations of $1, 2, \ldots, k$.

(2) Let $A_1, A_2, \ldots, A_k \in M_n$ be nonnegative matrices. Then for any $\tau, \gamma \in S_k$

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leq \rho^{\frac{1}{2}} \left(A_{\tau(1)} A_{\gamma(1)}^T \circ A_{\tau(2)} A_{\gamma(2)}^T \circ \cdots \circ A_{\tau(k)} A_{\gamma(k)}^T \right).$$

If k is even, then for any $\tau \in S_k$,

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leq \rho \left(A_{\tau(1)}^T A_{\tau(2)} \circ A_{\tau(3)}^T A_{\tau(4)} \circ \cdots \circ A_{\tau(k-1)}^T A_{\tau(k)} \right)$$

By specifying suitable permutations in (2), it can generalize some useful inequalities, which can refine some results due to Huang [5]. As an application, it implies that for nonnegative matrices $A, B \in M_n$,