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Linear maps preserving Kronecker quotients

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ABSTRACT

We study linear preserver problems connected to the Kronecker quotients. Characterizations of the linear preservers of uniform Kronecker quotients are obtained and examples of such linear preservers are given.

1. Introduction

Let m, n be positive integers and M_n the algebra of all $n \times n$ matrices over the field $\mathbb{F}$. If $A = [a_{ij}] \in M_m$ and $B \in M_n$, then the Kronecker product $A \otimes B$ is an $mn \times mn$ matrix

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$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix} \in M_{mn}.$$

From the block matrix form it is obvious that B can be determined from matrices A and $A \otimes B$ provided that A is nonzero. Methods for determining B from A and $A \otimes B$ have been described in the literature (see [1] and references therein). Let $0_n \in M_n$ denote the $n \times n$ zero matrix.

Definition 1.1. [1, Definition 2.2] A right Kronecker quotient is an operation $\oslash$, where $C \oslash B \in M_m$ when $C \in M_{mn}$ and $B \in M_n \setminus \{0_n\}$, which obeys

$$(A \otimes B) \oslash B = A$$

for all $A \in M_m$, $B \in M_n \setminus \{0_n\}$ and $m, n \in \mathbb{N}$.

From the above, a definition of the left Kronecker quotient $\obslash$ can be self-explanatory. In the remainder of the paper we only consider the right Kronecker quotient since analogous results for the left Kronecker quotient are straightforward. Note also that $B \oslash B = (1 \otimes B) \oslash B = 1$ for all non-zero $B \in M_n$.

Let us introduce the historical background of Kronecker quotients. First, in 1993, Van Loan and Pitsianis [2] described how to find A (for given C and B) which minimizes the Frobenius norm $\|C - (A \otimes B)\|_F$. In particular, they provided an example of a Kronecker quotient. More than ten years later, in 2005, Leopardi [3] introduced the notion of a Kronecker quotient $\obslash$ as a binary operation which is an inverse operation to the Kronecker matrix product (see the definition above) and considered some of its algebraic properties. Although these two Kronecker quotients were formulated differently, they share many algebraic properties. The reader is referred to the manuscript [1] for a comprehensive account of the subject.

For matrices $A \in M_m$, $B \in M_n$, let A^T and $\mathrm{tr}(A)$ denote the transpose of A and the trace of A, respectively, and let $\mathrm{tr}_2(A \otimes B)$ denote the partial trace over M_n in $M_{mn} = M_m \otimes M_n$, i.e., $\mathrm{tr}_2(A \otimes B) = \mathrm{tr}(B)A$. If $C \in M_{mn}$, then the partial Frobenius product $\bullet$ of C and B is given by

$$C \bullet B = \mathrm{tr}_2((I_m \otimes B^T)C).$$

In [1], Hardy introduced a uniform Kronecker quotient $\oslash$ as follows. A Kronecker quotient $\oslash$ is linear, if $(B \neq 0_n)$

(i) $(C_1 + C_2) \oslash B = C_1 \oslash B + C_2 \oslash B$ for all $C_1, C_2 \in M_{mn}$, $B \in M_n$;
(ii) $(\lambda C) \oslash B = \lambda(C \oslash B)$ for all $C \in M_{mn}$, $B \in M_n$, $\lambda \in \mathbb{F}$.