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Operator Ky Fan type inequalities

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Dedicated to the memory of Professor Ky Fan

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In this paper, we extend some significant Ky Fan type inequalities in a large setting to operators on Hilbert spaces and derive their equality conditions. Among other things, we prove that if $f : [0, \infty) \to [0, \infty)$ is an operator monotone function with $f(1) = 1$, $f'(1) = \mu$, and associated mean σ, then for all operators A and B on a complex Hilbert space $\mathscr{H}$ such that $0 < A, B \leq \frac{1}{2}I$, we have

$$A'\nabla_\mu B' - A'\sigma B' \leq A\nabla_\mu B - A\sigma B,$$

where I is the identity operator on $\mathscr{H}$, $A' := I - A$, $B' := I - B$, and ∇_μ is the μ-weighted arithmetic mean.

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1. Introduction

Let $n \geq 2$, and let $\mu_1, \ldots, \mu_n \geq 0$ such that $\sum_{i=1}^n \mu_i = 1$. For arbitrary real numbers $x_1, \ldots, x_n > 0$, we denote by A_n, G_n, and H_n the arithmetic mean, the geometric mean, and the harmonic mean of $x_1, \cdots, x_n$, respectively; that is,

$$A_n = \sum_{i=1}^n \mu_i x_i, \qquad G_n = \prod_{i=1}^n {x_i}^{\mu_i}, \qquad H_n = \frac{1}{\sum_{i=1}^n \mu_i \frac{1}{x_i}}.$$

For $x_i \in (0, \frac{1}{2}]$, we denote by A'_n, G'_n, and H'_n the arithmetic, geometric, and harmonic means of $x'_1 := 1 - x_1, \cdots, x'_n := 1 - x_n$, respectively; that is,

$$A'_n = \sum_{i=1}^n \mu_i x'_i, \qquad G'_n = \prod_{i=1}^n {x'_i}^{\mu_i}, \qquad H'_n = \frac{1}{\sum_{i=1}^n \mu_i \frac{1}{x'_i}}.$$

The most important and elegant Ky Fan type inequalities can be divided in the following three classes:

- **Additive class:**

$$A'_n - G'_n \leq A_n - G_n, \qquad \text{(Alzer)} \tag{1.1}$$

$$A'_n - H'_n \leq A_n - H_n. \qquad \text{(Alzer)} \tag{1.2}$$

- **Reciprocal additive class:**

$$\frac{1}{G'_n} - \frac{1}{A'_n} \leq \frac{1}{G_n} - \frac{1}{A_n}, \tag{1.3}$$

$$\frac{1}{H'_n} - \frac{1}{G'_n} \leq \frac{1}{H_n} - \frac{1}{G_n}, \tag{1.4}$$

$$\frac{1}{H'_n} - \frac{1}{A'_n} \leq \frac{1}{H_n} - \frac{1}{A_n}. \tag{1.5}$$

- **Multiplicative class:**

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}, \qquad \text{(Ky Fan)} \tag{1.6}$$

$$\frac{G'_n}{H'_n} \leq \frac{G_n}{H_n}, \qquad \text{(Wang–Wang)} \tag{1.7}$$

$$\frac{A'_n}{H'_n} \leq \frac{A_n}{H_n}. \tag{1.8}$$