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Binary Parseval frames from group orbits ☆

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Binary Parseval frames share many structural properties with real and complex ones. On the other hand, there are subtle differences, for example that the Gramian of a binary Parseval frame is characterized as a symmetric idempotent whose range contains at least one odd vector. Here, we study binary Parseval frames obtained from the orbit of a vector under a group representation, in short, binary Parseval group frames. In this case, the Gramian of the frame is in the algebra generated by the right regular representation. We identify equivalence classes of such Parseval frames with binary functions on the group that satisfy a convolution identity. This allows us to find structural constraints for such frames. We use these constraints to catalogue equivalence classes of binary Parseval frames obtained from group representations. As an application, we study the performance of binary Parseval frames generated with abelian groups for purposes of error correction. We show that if p is an odd prime, then the group $\mathbb{Z}_p^q$ is always preferable to $\mathbb{Z}_{p^q}$ when searching for best performing codes associated with binary Parseval group frames.

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1. Introduction

A binary frame is, in short, a finite, spanning family in a vector space over the Galois field with two elements. Binary frames share many properties with finite real or complex frames, which have been studied extensively in mathematics and engineering [9,16,17,6]. Because of the spanning property, frames can serve to expand any given vector in a linear combination of the frame vectors. In contrast to bases of a vector space, the frame vectors can include linear dependencies. If this is the case, then the expansion of a vector is no longer uniquely determined. However, for Parseval frames, there is a standard choice of coefficients appearing in the expansion of a vector that can be calculated efficiently. When the appropriate definition of a Parseval frame is made, then this holds in the binary as well as the real and complex setting [5]. In the case of real or complex frames, the expansion coefficients are computed by taking inner products with the frame vectors. The same property for binary frames requires replacing the inner product with the less restrictive concept of a bilinear form [14], which can be taken to be the standard dot product.

Equivalence classes are useful to classify frames and to study essential properties. In the real or complex case, a number of equivalence relations have been used, ranging from similarity for frames to unitary equivalence [11], projective unitary equivalence [8], and switching equivalence [10,4] for Parseval frames. As for real and complex frames, each set of unitarily equivalent binary Parseval frames can be identified with a corresponding Gramian [5, Proposition 4.8]. Even with this reduction up to unitary equivalence, the number of (inequivalent) binary Parseval frames appears to grow quickly as the dimension of the vector space and the number of frame vectors increase, see exhaustive lists for lowest dimensions in [5] and [1].

Next to similarities, binary frames exhibit differences with the theory of real and complex frames. One of the more striking ways in which they differ is when considering the Gram matrices. The Gram matrices of real or complex Parseval frames are characterized as symmetric or Hermitian idempotent matrices. In the binary case, these properties have to be augmented with the condition of at least one non-zero diagonal entry of the Gram matrix [1]. Equivalently, one of its column vectors must be odd, meaning it contains an odd number of 1's. The underlying reason is the range of the Gram matrix of a Parseval frame consists precisely of its eigenspace corresponding to eigenvalue one, which necessarily contains only odd vectors. If none of the column vectors were odd, then the span could not satisfy this requirement.

In this paper, we continue comparing the structure of binary Parseval frames with their real or complex counterparts. We specialize to frames obtained from the orbit of a vector under a group representation. There is already a substantial amount of literature on real and complex group frames [11,27,12,28,7,29]. Here, we study binary Parseval group frames, with special emphasis on the structure of the Gramians associated with them. Our first main result is that a binary Parseval frame is obtained from the action of a group if and only if its Gramian is in the group algebra.