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A characterization of oriented hypergraphic Laplacian and adjacency matrix coefficients

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ABSTRACT

An oriented hypergraph is an oriented incidence structure that generalizes and unifies graph and hypergraph theoretic results by examining its locally signed graphic substructure. In this paper we obtain a combinatorial characterization of the coefficients of the characteristic polynomials of oriented hypergraphic Laplacian and adjacency matrices via a signed hypergraphic generalization of basic figures of graphs. Additionally, we provide bounds on the determinant and permanent of the Laplacian matrix, characterize the oriented hypergraphs in which the upper bound is sharp, and demonstrate that the lower bound is never achieved.

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1. Introduction

Sachs' Coefficient Theorem provides a combinatorial interpretation of the coefficients of the characteristic polynomial of the adjacency matrix of a graph as families of subgraphs [9], generalizations and variants of this have been studied in [1–3,14,16]. In this paper we obtain an oriented hypergraphic generalization of Sachs' Coefficient Theorem that extends to the oriented hypergraphic Laplacian and the signless Laplacian. This extension shows that the standard adjacency matrix coefficients are the restricted enumeration of a family of sub-incidence-structures associated to any finite integral incidence matrix — providing a single class of combinatorial objects to study the coefficients of both characteristic polynomials.

These theorems are unified and generalized by using the weak walk Theorem for oriented hypergraphs in [13,6], which unifies the entries of the oriented hypergraphic matrices as weak walk counts, then constructing incidence preserving maps from disjoint 1-paths into a given oriented hypergraph. Restrictions of these maps to adjacency preserving maps on sub-oriented-hypergraphs obtained by weak deletion of vertices allows for the reclaiming of basic figures of graphs, as well as the determinant of the adjacency matrix by cycle covers.

The necessary oriented hypergraphic background and Sachs' Theorem are collected in Section 2. In Section 3 we examine the relationship between incidence preserving maps, weak walks, and generalized cycle covers called *contributors*. Section 4 establishes the permanent and determinant of the adjacency and Laplacian matrices as contributor counts as well as the main coefficient theorems of determinant and permanent versions of the characteristic polynomials. Finally, contributor counts are used to provide upper and lower bounds for the determinant and permanent of the Laplacian matrix over all orientations of a given underlying hypergraph. The lower bound is shown to never be sharp, while the family of oriented hypergraphs that achieve the upper bound are characterized.

2. Background

2.1. Oriented hypergraphs

These definitions are an adaptation of those appearing in [15,13,6], and allow for oriented hypergraphs to be treated as locally signed graphic through its adjacencies.

An *oriented hypergraph* is a quintuple $(V, E, \mathcal{I}, \iota, \sigma)$ where V, E, and $\mathcal{I}$ denote disjoint sets of *vertices*, *edges*, and *incidences*, respectively, with incidence function $\iota : \mathcal{I} \to V \times E$, and orientation function $\sigma : \mathcal{I} \to \{+1, -1\}$. We say v *and* e *are incident along* i if $\iota(i) = (v, e)$. Two incidences i and j are said to be *parallel* if $\iota(i) = \iota(j)$ — this provides an equivalence class of parallel incidences where the size of each equivalence class is called the *multiplicity of the incidence*.