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Diagonally singularizable matrices

Jiri Rohn [1]

Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic

A R T I C L E I N F O

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Dedicated to Professor Ilja Černý on the occasion of his 90th birthday

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A square matrix A is called diagonally singularizable if $|A-S| \leq I$ holds for some singular matrix S (I is the identity matrix). The paper brings several necessary and/or sufficient conditions for diagonal singularizability and demonstrates another specific features, namely existence of diagonal-singularizability-preserving operations and a theorem of symmetric alternative.

1. Introduction

In [1], the authors introduced the following concept: a matrix $A \in \mathbb{R}^{n\times n}$ is called *diagonally singularizable* if there exists a singular matrix S satisfying

$$|A - S| \leq I,$$

E-mail address: rohn@cs.cas.cz.
URL: http://uivtx.cs.cas.cz/~rohn.

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where I denotes the identity matrix and absolute value and inequality are understood entrywise. This term was used in [1] for formulating a nontrivial assertion (see Theorem 5 below): for each nonsingular matrix A, either A or A^{-1} is diagonally singularizable. It was just this remarkable property that prompted this author to investigate the concept of diagonal singularizability in more detail, thus giving rise to the present paper which brings several necessary and/or sufficient conditions for diagonal singularizability and demonstrates another specific features, namely existence of diagonal-singularizability-preserving operations and a theorem of symmetric alternative.

We use the following notation. $\varrho(A)$ stands for the spectral radius of A and $\lambda_{\min}(A)$ denotes the minimum eigenvalue of a symmetric matrix A. Let us recall that by the Courant–Fischer theorem [2],

$$\lambda_{\min}(A) = \min_{x \neq 0} \frac{x^T A x}{x^T x}.$$

Continuity of the minimum eigenvalue follows from the Wielandt–Hoffman theorem (see [2]):

$$|\lambda_{\min}(A) - \lambda_{\min}(B)| \leq \|A - B\|_F$$

holds for any two symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, where we use the Frobenius matrix norm $\|C\|_F = (\sum_{ij} c_{ij}^2)^{1/2}$. An interval matrix is a set of matrices of the form

$$[A - D,\ A + D] = \{\, C \mid |A - C| \leq D \,\}$$

with $D \geq 0$; it is called singular if it contains a singular matrix. For a $t \in \mathbb{R}^n$, T_t denotes the diagonal matrix with diagonal vector t. $\{-1, 1\}^n$ is the set of all ± 1-vectors in $\mathbb{R}^n$ (there are 2^n of them). Let us note that $|AB| \leq |A||B|$ whenever the matrices A, B can be multiplied.

2. Necessary and sufficient conditions

First, we have several necessary and sufficient conditions for diagonal singularizability.

Theorem 1. *For a matrix $A \in \mathbb{R}^{n \times n}$, the following assertions are equivalent:*

(i) *A is diagonally singularizable,*
(ii) *$[A - I,\ A + I]$ is singular,*
(iii) *$|Ax| \leq |x|$ for some $x \neq 0$,*
(iv) *$\det(A)\det(A - T_y) \leq 0$ for some $y \in \{-1, 1\}^n$,*
(v) *$A - \tau T_y$ is singular for some $\tau \in [0, 1]$ and $y \in \{-1, 1\}^n$,*
(vi) *$|Ax| = \tau |x|$ for some $\tau \in [0, 1]$ and $x \neq 0$.*