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Matrices over finite fields as sums of periodic and nilpotent elements

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ABSTRACT

We prove that every $n \times n$ matrix M over a field of odd cardinality q has a decomposition of the form $M = E + N$ such that $E^q = E$ and $N^3 = 0$.

1. Introduction

The study of representing matrices over division rings as sums of nilpotent and idempotent matrices has a long history. We refer to [5], [7], and [12] for some classical results, respectively to [2], [4], [8], [11] and [9] for some recent approaches.

In [3] and [6] it was proved that every matrix from the ring $\mathcal{M}_n(D)$ of n-by-n matrices over a division ring D is a sum of an idempotent matrix and a nilpotent matrix if and

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only if $D = \mathbb{F}_2$, the field with two elements. This result was substantially improved by Šter in [13, Theorem 2]. He proved that every matrix M over $\mathbb{F}_2$ has a decomposition of the form $M = E + N$ such that E is idempotent and N a nilpotent matrix of nilpotence degree at most 4 (i.e. $N^4 = 0$).

The main result of [3] was extended in [1] to finite fields of arbitrarily characteristic: if F is a field of cardinality q then every matrix over F has a decomposition $M = E + N$ such that $E^q = E$ and N is nilpotent. In this note we will prove that if the field F is of odd characteristic then there exists such a decomposition such that the nilpotence degree of N is at most 3:

Theorem 1. *Let F be a finite field of odd cardinality q. If n positive integer and M is an $n \times n$ matrix over F then there exist two $n \times n$ matrices E and N such that $M = E + N$, $E^q = E$ and $N^3 = 0$.*

In the end it is proved that there exist 3×3 matrices A over $\mathbb{F}_3$ which cannot be decomposed as $A = E + N$ with $E^3 = E$ and $N^2 = 0$.

2. The proof

We fix a finite field F of odd cardinality q. It is enough to prove that every companion matrix

$$C_{c_0,c_1,\dots,c_{n-1}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_{n-2} \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

has a decomposition as those described in Theorem 1.

We will use the following result, which is extracted from (the proof of) [13, Lemma 2.1]:

Lemma 2. *Let F be a field. For every companion matrix $C = C_{c_0,c_1,\dots,c_{n-1}}$ there exists a basis $(f_1, \dots, f_n)$ of F^n such that*

1. *the matrix associated to C with respect to $(f_1, \dots, f_n)$ is of the form*

$$D_{d_0,\dots,d_{n-1}} = C_{d_0,d_1,\dots,d_{n-1}} + \mathrm{diag}(1, 0, 1, 0, \dots).$$

2. *$f_1 = e_1 = (1, 0, \dots, 0)$, the first vector from the canonical basis of F^n.*

We will use this lemma to conclude that it is enough to consider matrices of some special forms. In order to describe these forms, we will use the notations: