Linear Algebra and its Applications 555 (2018) 98–138

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Equiangular tight frames that contain regular simplices

Matthew Fickus[a,*], John Jasper[b], Emily J. King[c], Dustin G. Mixon[d]

[a] *Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson AFB, OH 45433, United States of America*
[b] *Department of Mathematics and Statistics, South Dakota State University, Brookings, SD 57007, United States of America*
[c] *Department of Mathematics and Computer Science, University of Bremen, Bremen, 28359, Germany*
[d] *Department of Mathematics, Ohio State University, Columbus, OH 43210, United States of America*

A R T I C L E   I N F O

Article history:
Received 19 November 2017
Accepted 2 June 2018
Available online 13 June 2018
Submitted by H. Rauhut

MSC:
42C15

Keywords:
Equiangular
Tight
Frame

A B S T R A C T

An equiangular tight frame (ETF) is a type of optimal packing of lines in Euclidean space. A regular simplex is a special type of ETF in which the number of vectors is one more than the dimension of the space they span. In this paper, we consider ETFs that contain a regular simplex, that is, have the property that a subset of its vectors forms a regular simplex. As we explain, such ETFs are characterized as those that achieve equality in a certain well-known bound from the theory of compressed sensing. We then consider the so-called binder of such an ETF, namely the set of all regular simplices that it contains. We provide a new algorithm for computing this binder in terms of products of entries of the ETF's Gram matrix. In certain circumstances, we show this binder can be used to produce a particularly elegant Naimark complement of the corresponding ETF. Other times, an ETF is a disjoint union of regular simplices, and we show this leads to a certain type of optimal packing of subspaces known as an equichordal tight fusion frame. We conclude by considering

* Corresponding author.
E-mail address: Matthew.Fickus@gmail.com (M. Fickus).

https://doi.org/10.1016/j.laa.2018.06.004
0024-3795/Published by Elsevier Inc.

the extent to which these ideas can be applied to numerous known constructions of ETFs, including harmonic ETFs.

1. Introduction

Let n and d be positive integers with $n \geq d$, and let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. The *coherence* of a sequence $\{\boldsymbol{\varphi}_j\}_{j=1}^n$ of n nonzero equal norm vectors in a d-dimensional Hilbert space $\mathbb{H}$ over $\mathbb{F}$ is

$$\mu := \max_{j \neq j'} \tfrac{|\langle \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_{j'} \rangle|}{\|\boldsymbol{\varphi}_j\| \|\boldsymbol{\varphi}_{j'}\|}. \tag{1}$$

In the real case, each vector $\boldsymbol{\varphi}_j$ spans a line and μ is the cosine of the smallest angle between any pair of these lines. Our work here is motivated by two well-known bounds involving μ. The first of these is the *Welch bound* [62], which is a lower bound on μ whenever $n \geq d$:

$$\mu \geq \left[\tfrac{n-d}{d(n-1)}\right]^{\frac{1}{2}}. \tag{2}$$

The second bound arises in compressed sensing [28,18], and gives a lower bound on the *spark* of $\{\boldsymbol{\varphi}_j\}_{j=1}^n$, namely the smallest number of these vectors that are linearly dependent:

$$\operatorname{spark}\{\boldsymbol{\varphi}_j\}_{j=1}^n \geq \tfrac{1}{\mu} + 1. \tag{3}$$

It is well known that $\{\boldsymbol{\varphi}_j\}_{j=1}^n$ achieves equality in the Welch bound (2) if and only if it is an *equiangular tight frame* (ETF) for $\mathbb{H}$, that is, if and only if the value of $|\langle \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_{j'} \rangle|$ is constant over all $j \neq j'$ (equiangularity) and there exists $a > 0$ such that $\sum_{j=1}^n |\langle \boldsymbol{\varphi}_j, \mathbf{x} \rangle|^2 = a\|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{H}$ (tightness) [57]. This paper focuses on ETFs that achieve equality in (3). As we shall see, this happens precisely when the ETF contains a *regular simplex*, namely when for some positive integer s there are $s+1$ of the $\boldsymbol{\varphi}_j$ vectors that form an ETF for an s-dimensional subspace of $\mathbb{H}$. Of the few infinite families of ETFs that are known, a remarkably large proportion of them have this property. This raises the following fundamental question: in general, to what extent do ETFs contain regular simplices? Our results here are some first steps towards an answer.

ETFs arise in several applications including waveform design for wireless communication [57], compressed sensing [5,8], quantum information theory [64,54] and algebraic coding theory [45]. They also seem to be rare [34]. With the exception of orthonormal bases and regular simplices, every known infinite family of ETFs involves some type of combinatorial design. Real ETFs are equivalent to a subclass of strongly regular graphs (SRGs) [50,55,43,61], and such graphs have been actively studied for decades [16,17,24].