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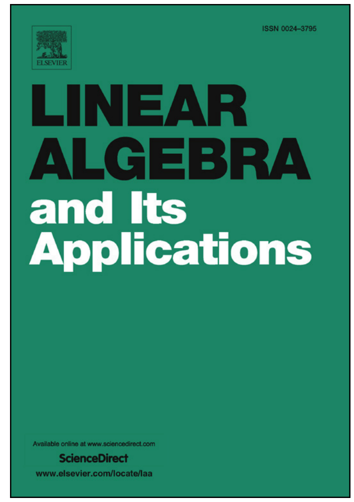

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Joint numerical ranges of operators in semi-Hilbertian spaces

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Abstract

In this paper we aim to investigate the concept of numerical range and maximal numerical range relative to a positive operator of a d-tuple of bounded linear operators on a Hilbert space. Some properties and applications of these sets are studied. Mainly, it is proved that they are convex for $d = 1$, this generalizes the well known Toeplitz-Hausdorff Theorem [24, 16] and Stampi's result [23]. Moreover, under additional hypotheses, we show that these sets are convex for $d \geq 2$.

Keywords. Positive operator, joint numerical range, maximal numerical range, convexity.
Mathematics Subject Classification Primary 46C05; Secondary 47A05.

1 Introduction and Preliminaries

Let $\mathcal{H}$ be a non trivial complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ and associated norm $\| \cdot \|$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the classical numerical range of T was introduced by Toeplitz in [24] as

$$W(T) := \{\langle Tx \mid x\rangle;\ x \in \mathcal{H} \text{ with } \|x\| = 1\}.$$

By the Toeplitz-Hausdorff Theorem, $W(T)$ is convex. This theorem has many proofs, a recent one is due to C.K. Li [18]. A short proof covering unbounded operators acting on a pre-Hilbert space was given by K. Gustafson [14]. Other basic properties of $W(T)$ can be found in [5]. There is a rich variety of generalizations of the notion of the numerical range. For example in [13], W. Givens has introduced the generalized numerical range of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ by considering the generalized inner product induced by a positive definite hermitian matrix N. It is defined by

$$F_N(M) := \{x^*NMx;\ x \in \mathbb{C}^n,\ x^*Nx = 1\}.$$

The concept of the classical numerical range was generalized to the joint numerical range by A.T. Dash [10] as follows.

Definition 1.1. *([10]) Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a d-tuple of operators. The joint numerical range of $\mathbf{T}$ is the subset of $\mathbb{C}^d$ defined by*

$$JtW(\mathbf{T}) = \{(\langle T_1x \mid x\rangle, \cdots, \langle T_dx \mid x\rangle);\ x \in \mathcal{H},\ \|x\| = 1\}.$$