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An upper bound of the nullity of a graph in terms of order and maximum degree

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ABSTRACT

Let G be a finite undirected graph without loops and multiple edges. By η, Δ and n we respectively denote the nullity, the maximum vertex degree and the order of G. In [10], it was proved that $\eta \leq n - 2\lceil \frac{n-1}{\Delta} \rceil$ when G is a tree. This result was generalized to a bipartite graph by [25]. For a reduced bipartite graph G, the above inequality was improved to $n - 2 - 2\ln_2\Delta$ by [27]. However, the problem of bounding the nullity of an arbitrary graph G in terms of n and Δ is left open for more than ten years. In this article, we aim to solve such a left problem. We prove that $\eta \leq \frac{\Delta-1}{\Delta}n$ for an arbitrary graph G with order n and maximum degree Δ, and the equality holds if and only if G is the disjoint union of some copies of $K_{\Delta,\Delta}$.

1. Introduction

The chemical importance of the nullity of a graph lies in the fact, that within the Hückel molecular orbital model, if $\eta(G) > 0$ for the molecular graph G, then the cor-

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responding chemical compound is highly reactive and unstable, or nonexistent (see [1] or [7]).

In 1957, Collatz and Sinogowitz [6] posed the problem of characterizing all singular graphs. The problem has received considerable attention in the literature (see for example [2–4,9–11,13–20,23,25–30,32] and references therein). In studying the above problem, some authors focused on describing the nullity set of certain classes of graphs in terms of the order of graphs. Let $\mathcal{G}_n$ be the set of a class of graphs of order n, and let $[0, n] = \{0, 1, 2, ..., n\}$. A subset N of $[0, n]$ is said to be the nullity set of $\mathcal{G}_n$ if $\eta(G) \in N$ for each $G \in \mathcal{G}_n$, and for each $k \in N$ there exists at least one graph $G \in \mathcal{G}_n$ such that $\eta(G) = k$. Some known results about the nullity sets of certain families of graphs are listed as follows.

- The nullity set of unicyclic graphs of order $n \geq 5$ is $[0, n-4]$ (see [28]).
- The nullity set of bicyclic graphs of order n is $[0, n-2]$ (see [16]).
- The nullity set of tricyclic graphs of order $n \geq 8$ is $[0, n-4]$ (see [4]).
- The nullity set of bipartite graphs of order n is $\{n-2k : k = 0, 1, 2, ..., \lfloor n/2 \rfloor\}$ (see [9] or [25]).
- If T is a tree, then the nullity of its line graph is either 0 or 1 (see [14]).
- Denote by C_k and L_k the set of all connected graphs with k induced cycles and the set of line graphs of all graphs of C_k. For any given integer k, the nullity set of L_k is $[0, k+1]$ (see [11]).

Some attention are also attracted to bound the nullity of a graph by using other structure parameters, such as the matching number, the number of pendant vertices, especially the maximum degree of the graph (see below).

Using the matching number to bound the nullity of a graph is initiated by Cvetković in [7]. The author in [7] proved that $\eta(G) = n - 2m$ for a bipartite graph G of order n which does not contain any cycle of length a multiple of 4, where m is the matching number of G. Guo et al. [13] bounded the nullity of a unicyclic graph in terms of the matching number by establishing the following inequality

$$n - 2m - 1 \leq \eta(G) \leq n - 2m + 2.$$

In [29], Wang et al. generalized the above equality to an arbitrary graph and obtained an inequality:

$$n - 2m - \theta \leq \eta(G) \leq n - 2m + 2\theta,$$

where θ is the dimension of cycle space of G. Ma et al. [23] established an upper bound $\eta(G) \leq 2\theta + p$ by using θ and p, where p is the number of pendant vertices of G. More recently, some other publications (see [5], [21], [22], [24], [29], [31]) focused on studying the skew-rank of oriented graphs or H-rank of mixed graphs.