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A congruence on the semiring of normal tropical matrices

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We introduce and study a congruence ρ on the normal tropical matrix semiring M_n^N, which is relevant to the Kleene stars of normal tropical matrices. We prove that this congruence is a bisemilattice congruence and give an exact description of each ρ-class. In particular, we show that the ρ-class $E\rho$ is an interval when E is a strongly regular normal tropical matrix. We also present a method using Floyd–Warshall algorithm to compute the greatest lower bound of $E\rho$ in such case.

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1. Introduction and preliminaries

By a *semiring* we mean a nonempty set S with two binary operations $\oplus$ (addition) and $\otimes$ (multiplication) such that

- $(S, \oplus)$ is a commutative semigroup;
- $(S, \otimes)$ is a semigroup;
- $\otimes$ distributes over $\oplus$.

A semiring S is said to be *commutative* if $a \otimes b = b \otimes a$ for all $a, b \in S$. An element e (ε, respectively) of a semiring S is called a *neutral element for multiplication* (a *neutral element for addition*, respectively) if $a \otimes e = e \otimes a = a$ ($a \oplus \varepsilon = a$, respectively) for all $a \in S$. A semiring with a multiplicative neutral element e and an additive neutral element ε is called an *idempotent semiring* (also called *dioid*, see [1,11,19]) if $a \oplus a = a$ for all $a \in S$. Notice that some authors (see [9,10] for instance) require that the additive neutral element ε is absorbing. However we do not require our semirings and idempotent semirings to satisfy this property. If we define the binary relation $\leq$ on an idempotent semiring S by

$$a \leq b \Longleftrightarrow a \oplus b = b, \tag{1.1}$$

then it is a partial order relation (see [1]). Also, it is compatible with $\oplus$ and $\otimes$, that is, for any $a, b, c \in S$,

$$a \leq b \Rightarrow a \oplus c \leq b \oplus c,\ a \otimes c \leq b \otimes c \text{ and } c \otimes a \leq c \otimes b. \tag{1.2}$$

The tropical semiring $\overline{\mathbb{R}}$ is a commutative idempotent semiring (that is, a commutative dioid). It is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of tropical addition $a \oplus b = \max(a, b)$ and tropical multiplication $a \otimes b = a + b$, where 0 and $-\infty$ are the multiplicative neutral element and the additive neutral element, respectively. Note that the partial order $\leq$ on $\overline{\mathbb{R}}$ defined by (1.1) is the usual order of real numbers. Tropical algebra (also called max-plus algebra) is the algebra developed over the tropical semiring. It provides an attractive way to describe and solve nonlinear problems appearing in areas such as combinatorial optimization [3], control theory [5], geometry [8,15], automata theory [21], etc. Many such problems are naturally expressed using systems of tropical linear equations and inequalities, and so tropical matrices have been intensively studied (see [1,3–5,12,15,16,21]).

As in conventional linear algebra, we can extend the operations $\oplus$ and $\otimes$ on the tropical semiring $\overline{\mathbb{R}}$ to matrices. That is, if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are matrices over $\overline{\mathbb{R}}$ of compatible sizes, then we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj}$ for all i, j. With respect to the tropical matrix addition $\oplus$ and the tropical matrix multiplication $\otimes$, the set M_n of all $n \times n$ tropical