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# P-tensors, $P_0$-tensors, and their applications ☆

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P- and $P_0$-matrix classes have wide applications in mathematical analysis, linear and nonlinear complementarity problems, etc., since they contain many important special matrices, such as positive (semi-)definite matrices, M-matrices, diagonally dominant matrices, etc. By modifying the existing definitions of P- and $P_0$-tensors that work only for even order tensors, in this paper, we propose a homogeneous formula for the definition of P- and $P_0$-tensors. The proposed P- and $P_0$-tensor classes coincide the existing ones of even orders and include many important structured tensors of odd orders. We show that many checkable classes of structured tensors, such as the nonsingular M-tensors, the nonsingular H-tensors with positive diagonal entries, the strictly diagonally dominant tensors with positive diagonal entries, are P-tensors under the new definition, regardless of whether the order is even or odd. In the odd order case, our definition of $P_0$-tensors, to some extent, can be regarded as an extension of positive semi-definite (PSD) tensors. The theoretical applications of P- and

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$P_0$-tensors under the new definition to tensor complementarity problems and spectral hypergraph theory are also studied.


## 1. Introduction

The P-matrix, first introduced by Fiedler and Pták [1], is an important type of a special matrix. It is a square matrix all of whose determinants of its principal submatrices are positive. The class of P-matrices contains many notable matrices as its special cases, such as positive definite matrices, nonsingular M-matrices, nonsingular completely positive matrices, strictly diagonally dominant matrices, etc. In a certain sense, P-matrices and $P_0$-matrices can be regarded as extensions of positive definite symmetric matrices and positive semi-definite symmetric matrices to the nonsymmetric case, since a symmetric matrix is positive definite (or positive semidefinite respectively) if and only if it is a P-matrix (or $P_0$-matrix respectively). Besides its significance in matrix analysis, the P-matrix is also important for linear complementarity problems (see, e.g., [2–5]). Given a matrix $A$ and a vector $\mathbf{q}$, a linear complementarity problem LCP($\mathbf{q}, A$) is to find a vector $\mathbf{x}$ such that

$$\mathbf{x} \geq \mathbf{0}, \;\; A\mathbf{x} + \mathbf{q} \geq \mathbf{0}, \;\; \langle \mathbf{x}, A\mathbf{x} + \mathbf{q} \rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. The corresponding LCP($\mathbf{q}, A$) has a unique solution for any vector $\mathbf{q}$ if and only if the matrix $A$ is a P-matrix. P-matrices are also widely applied in nonlinear complementarity problems. For instance, Qi, Sun, and Zhou [6] employed some properties of P-matrices to investigate the convergence of smoothing Newton methods for nonlinear complementarity problems.

With an emerging interest in multi-linear algebra concentrated on the higher-order tensors, more structured matrices have been generalized to higher-order cases. Here the tensor is referred to a hyper-matrix, or a multi-way array. In 2015, Song and Qi [7] extended P- and $P_0$-matrices to P- and $P_0$-tensors. In the even-order case, it was shown that a symmetric tensor is positive definite (PD) (positive semi-definite (PSD) respectively) if and only if it is a P-tensor ($P_0$-tensor respectively). The P- and $P_0$-tensors were shown to be applicable in the tensor complementarity problem ([7–10]) which is referred to finding some vector $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\mathbf{x} \geq \mathbf{0}, \;\; \mathcal{A}\mathbf{x}^{m-1} + \mathbf{q} \geq \mathbf{0}, \;\; \langle \mathbf{x}, \mathcal{A}\mathbf{x}^{m-1} + \mathbf{q} \rangle = 0, \tag{2}$$

where $\mathcal{A} = (a_{i_1 \cdots i_m})$ is an $m$th-order $n$-dimensional tensor,

$$\mathcal{A}\mathbf{x}^{m-1} := \left( \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right) \in \mathbb{R}^n$$