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Equality of three numerical radius inequalities

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ABSTRACT

For an n-by-n matrix A, let $w(A)$ and $\|A\|$ denote its numerical radius and operator norm, respectively. The following three inequalities, each a strengthening of $w(A) \leq \|A\|$, are known to hold: $w(A)^2 \leq (\|A\|^2 + w(A^2))/2$, $w(A) \leq (\|A\| + \|A^2\|^{1/2})/2$, and $w(A) \leq (\|A\| + w(\Delta_t(A)))/2$ $(0 \leq t \leq 1)$, where $\Delta_t(A)$ is the generalized Aluthge transform of A. In this paper, we derive necessary and sufficient conditions in terms of the operator structure of A for which the inequalities become equalities.

1. Introduction

For an n-by-n (complex) matrix A, its *operator norm* $\|A\|$, *numerical range* $W(A)$, and *numerical radius* $w(A)$ are given by $\|A\| = \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}$, $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$, and $w(A) = \max\{|z| : z \in W(A)\}$, respectively, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^n$. It is easily seen, as a consequence of

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the Cauchy–Schwarz inequality $|\langle Ax, x\rangle| \le \|A\|\|x\|^2$ and its equality condition, that $w(A) \le \|A\|$ and $w(A) = \|A\|$ holds if and only if A is unitarily similar to a matrix of the form $[a] \oplus B$, where B is an $(n-1)$-by-$(n-1)$ matrix with $\|B\| \le |a|$. Over the years, this has been improved to various sharper inequalities. For example, Kittaneh showed in 2003 that

$$w(A) \le \frac{1}{2}\big(\|A\| + \|A^2\|^{1/2}\big) \tag{1.1}$$

holds (cf. [11, Theorem 1]). This is further refined to

$$w(A) \le \frac{1}{2}\Big(\|A\| + w\big(\Delta_{1/2}(A)\big)\Big)$$

by Yamazaki [13, Theorem 2.1] and to

$$w(A) \le \frac{1}{2}\Big(\|A\| + w\big(\Delta_t(A)\big)\Big) \tag{1.2}$$

for any t, $0 \le t \le 1$, by Abu Omar and Kittaneh [1, Theorem 3.2]. Here $\Delta_t(A)$ denotes, for $0 \le t \le 1$, the *generalized Aluthge transform* of A, that is,

$$\Delta_t(A) = |A|^t V |A|^{1-t}, \tag{1.3}$$

where $|A| = (A^*A)^{1/2}$ and V is the partial isometry defined on $\mathbb{C}^n = \operatorname{ran} A^* \oplus \ker A$ by $V(|A|x) = Ax$ for any vector x in $\mathbb{C}^n$ and $Vy = 0$ for y in $\ker A$. In fact, V is the (unique) partial isometry appearing in the *polar decomposition* $A = V|A|$ of A with $\ker V = \ker A$. Basic properties of the Aluthge transforms will be given in Section 4 below. In another vein, Dragomir gave in [6, Theorem 1] an upper estimate

$$w(A)^2 \le \frac{1}{2}\big(\|A\|^2 + w(A^2)\big) \tag{1.4}$$

of $w(A)$ by way of his (generalized) Cauchy–Schwarz inequality. However, in all these works, attention was rarely paid to when the equalities hold. In this paper, we address this problem by deriving necessary and sufficient conditions in terms of the operator structure of A for which the inequalities become equalities. Sections 2, 3 and 4 below deal with the cases of (1.4), (1.1) and (1.2), respectively. Note that the three inequalities are all true for bounded linear operators on a Hilbert space. In this paper, we restrict ourselves to finite matrices for the ease of exposition.

We use 0_n and I_n to denote the n-by-n zero matrix and identity matrix, respectively. For an n-by-n matrix A, $\operatorname{Re} A = (A + A^*)/2$ is its real part, and $\operatorname{ran} A$ and $\ker A$ are its range and kernel, respectively. For a subset L of $\mathbb{C}^n$, $\bigvee L$ denotes the subspace spanned by the vectors in L. $M_n(\mathbb{C})$ is the space of all n-by-n complex matrices.

Our references for properties of the numerical range and numerical radius are [10, Chapter 1] and [8, Chapter 22].