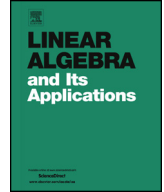




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Linear Algebra and its Applications

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A poset  $\Phi_n$  whose maximal chains are in bijection with the  $n \times n$  alternating sign matrices



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ARTICLE INFO

Article history:

Received 18 October 2017  
 Accepted 25 May 2018  
 Available online 28 May 2018  
 Submitted by R. Brualdi

MSC:

primary 05B20  
 secondary 05E18, 15B35, 15B36

Keywords:

Alternating sign matrix  
 Maximal chain  
 Dihedral group

ABSTRACT

For an integer  $n \geq 1$ , we display a poset  $\Phi_n$  whose maximal chains are in bijection with the  $n \times n$  alternating sign matrices. The Hasse diagram  $\widehat{\Phi}_n$  is obtained from the  $n$ -cube by adding some edges. We show that the dihedral group  $D_{2n}$  acts on  $\widehat{\Phi}_n$  as a group of automorphisms.

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1. Introduction

We will be discussing a type of square matrix called an alternating sign matrix. These matrices were introduced in [5], and subsequently linked to many other topics in Combinatorics; see [1] for an overview. In [4], the alternating sign matrices are linked to partially ordered sets in the following way. For  $n \geq 1$ , the set of  $n \times n$  alternating sign matrices becomes a distributive lattice, which is the MacNeille completion of the Bruhat order on the symmetric group  $S_n$ . For more discussion of this see [3], [6, p. 598], [7, p. 2].

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In the present paper we link alternating sign matrices to partially ordered sets in a different way. For  $n \geq 1$ , we display a partially ordered set  $\Phi_n$ , whose maximal chains are in bijection with the  $n \times n$  alternating sign matrices. We will also discuss the symmetries of  $\Phi_n$ . Before describing our results in more detail, we recall a few terms.

Consider a finite poset, with vertex set  $X$  and partial order  $\leq$ . For vertices  $x, y$  write  $x < y$  whenever  $x \leq y$  and  $x \neq y$ . A vertex  $x$  is *maximal* (resp. *minimal*) whenever there does not exist a vertex  $y$  such that  $x < y$  (resp.  $y < x$ ). For vertices  $x, y$  we say that  $y$  *covers*  $x$  whenever  $x < y$  and there does not exist a vertex  $z$  such that  $x < z < y$ . For an integer  $r \geq 0$ , a *chain of length  $r$*  is a sequence of vertices  $x_0 < x_1 < \dots < x_r$ . This chain is called *maximal* whenever (i)  $x_r$  is maximal; (ii)  $x_0$  is minimal; (iii)  $x_i$  covers  $x_{i-1}$  for  $1 \leq i \leq r$ . The *Hasse diagram*  $\widehat{X}$  is an undirected graph with vertex set  $X$ ; vertices  $x, y$  are adjacent in  $\widehat{X}$  whenever one of  $x, y$  covers the other one.

For the rest of this paper, fix an integer  $n \geq 1$ . An  $n \times n$  matrix is said to have order  $n$ .

We recall the  $n$ -cube. This is an undirected graph, whose vertex set consists of the sequences  $(a_1, a_2, \dots, a_n)$  such that  $a_i \in \{0, 1\}$  for  $1 \leq i \leq n$ . Vertices  $x, y$  of the  $n$ -cube are adjacent whenever they differ in exactly one coordinate. The  $n$ -cube is often called a *hypercube*, or a *binary Hamming graph*.

We now describe our results in more detail. Consider the poset whose vertex set consists of the subsets of  $\{1, 2, \dots, n\}$ ; the partial order is  $\subseteq$ . For this poset, the Hasse diagram is isomorphic to the  $n$ -cube and the maximal chains are in bijection with the permutation matrices of order  $n$  [2, p 142]. We augment this partial order by adding some edges to the Hasse diagram, and denote the resulting poset by  $\Phi_n$ . We show that the maximal chains in  $\Phi_n$  are in bijection with the alternating sign matrices of order  $n$ . We show that the dihedral group  $D_{2n}$  acts on  $\widehat{\Phi}_n$  as a group of automorphisms.

## 2. Alternating sign matrices

In this section we give some definitions and elementary facts concerning alternating sign matrices.

**Definition 2.1.** A sequence  $(\sigma_0, \sigma_1, \dots, \sigma_n)$  is *constrained* whenever

- (i)  $\sigma_i \in \{0, 1\}$  for  $0 \leq i \leq n$ ;
- (ii)  $\sigma_0 = 0$  and  $\sigma_n = 1$ .

Let  $\text{Con}_n$  denote the set of constrained sequences.

Given a constrained sequence  $(\sigma_0, \sigma_1, \dots, \sigma_n)$  define  $\alpha_i = \sigma_i - \sigma_{i-1}$  for  $1 \leq i \leq n$ . For example, if  $(\sigma_0, \sigma_1, \dots, \sigma_n) = (0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1)$  then the sequence  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  is

$$(0, 1, 0, 0, -1, 1, 0, -1, 0, 1).$$

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