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A poset Φ_n whose maximal chains are in bijection with the $n \times n$ alternating sign matrices

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For an integer $n \geq 1$, we display a poset Φ_n whose maximal chains are in bijection with the $n \times n$ alternating sign matrices. The Hasse diagram $\widehat{\Phi}_n$ is obtained from the n-cube by adding some edges. We show that the dihedral group D_{2n} acts on $\widehat{\Phi}_n$ as a group of automorphisms.

1. Introduction

We will be discussing a type of square matrix called an alternating sign matrix. These matrices were introduced in [5], and subsequently linked to many other topics in Combinatorics; see [1] for an overview. In [4], the alternating sign matrices are linked to partially ordered sets in the following way. For $n \geq 1$, the set of $n \times n$ alternating sign matrices becomes a distributive lattice, which is the MacNeille completion of the Bruhat order on the symmetric group S_n. For more discussion of this see [3], [6, p. 598], [7, p. 2].

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In the present paper we link alternating sign matrices to partially ordered sets in a different way. For $n \geq 1$, we display a partially ordered set Φ_n, whose maximal chains are in bijection with the $n \times n$ alternating sign matrices. We will also discuss the symmetries of Φ_n. Before describing our results in more detail, we recall a few terms.

Consider a finite poset, with vertex set X and partial order $\leq$. For vertices x, y write $x < y$ whenever $x \leq y$ and $x \neq y$. A vertex x is *maximal* (resp. *minimal*) whenever there does not exist a vertex y such that $x < y$ (resp. $y < x$). For vertices x, y we say that y *covers* x whenever $x < y$ and there does not exist a vertex z such that $x < z < y$. For an integer $r \geq 0$, a *chain of length* r is a sequence of vertices $x_0 < x_1 < \cdots < x_r$. This chain is called *maximal* whenever (i) x_r is maximal; (ii) x_0 is minimal; (iii) x_i covers x_{i-1} for $1 \leq i \leq r$. The *Hasse diagram* $\widehat{X}$ is an undirected graph with vertex set X; vertices x, y are adjacent in $\widehat{X}$ whenever one of x, y covers the other one.

For the rest of this paper, fix an integer $n \geq 1$. An $n \times n$ matrix is said to have order n.

We recall the n-cube. This is an undirected graph, whose vertex set consists of the sequences $(a_1, a_2, \ldots, a_n)$ such that $a_i \in \{0, 1\}$ for $1 \leq i \leq n$. Vertices x, y of the n-cube are adjacent whenever they differ in exactly one coordinate. The n-cube is often called a *hypercube*, or a *binary Hamming graph*.

We now describe our results in more detail. Consider the poset whose vertex set consists of the subsets of $\{1, 2, \ldots, n\}$; the partial order is $\subseteq$. For this poset, the Hasse diagram is isomorphic to the n-cube and the maximal chains are in bijection with the permutation matrices of order n [2, p 142]. We augment this partial order by adding some edges to the Hasse diagram, and denote the resulting poset by Φ_n. We show that the maximal chains in Φ_n are in bijection with the alternating sign matrices of order n. We show that the dihedral group D_{2n} acts on $\widehat{\Phi}_n$ as a group of automorphisms.

2. Alternating sign matrices

In this section we give some definitions and elementary facts concerning alternating sign matrices.

Definition 2.1. A sequence $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ is *constrained* whenever

(i) $\sigma_i \in \{0, 1\}$ for $0 \leq i \leq n$;
(ii) $\sigma_0 = 0$ and $\sigma_n = 1$.

Let Con_n denote the set of constrained sequences.

Given a constrained sequence $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ define $\alpha_i = \sigma_i - \sigma_{i-1}$ for $1 \leq i \leq n$. For example, if $(\sigma_0, \sigma_1, \ldots, \sigma_n) = (0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1)$ then the sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is

$$(0, 1, 0, 0, -1, 1, 0, -1, 0, 1).$$