Linear Algebra and its Applications 554 (2018) 329–341

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A new aspect of Riordan arrays via Krylov matrices ☆

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A R T I C L E   I N F O

Article history:
Received 22 November 2017
Accepted 27 May 2018
Available online 31 May 2018
Submitted by R. Brualdi

MSC:
15A30
05A15

Keywords:
Riordan array
Krylov matrix
Riordan group
Riordan Lie algebra

A B S T R A C T

In this paper, we give a new angle to interpret Riordan arrays by showing that every Riordan array can be expressed as a Krylov matrix. We then use this idea to obtain some groups containing the Riordan group as a subgroup. Moreover, we study Lie algebras for the extended Riordan groups as Lie groups.

1. Introduction

An $n \times n$ matrix is said to be the *Krylov matrix* of a matrix $A \in M_n(\mathbb{C})$ by a vector $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$ if its kth column is equal to $A^{k-1}\mathbf{v}$. We write $K(A, \mathbf{v})$ for the matrix:

☆ This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (2016R1A2B4011321 and 2016R1A5A1008055).

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https://doi.org/10.1016/j.laa.2018.05.028

$$K(A,\mathbf{v}) = \begin{bmatrix} \mathbf{v} & A\mathbf{v} & A^2\mathbf{v} & \cdots & A^{n-1}\mathbf{v} \end{bmatrix} \in M_n(\mathbb{C}). \tag{1}$$

Krylov matrices play a fundamental role in matrix computations. They are a key tool in understanding and developing numerical methods for solving eigenvalue problems of large sparse matrices and systems of linear equations.

If A is a triangular matrix then the Krylov matrix $K(A,\mathbf{v})$ might have special underlying structures. For example, if A is a unit lower triangular matrix of the form:

$$A = \begin{bmatrix} 1 & & & \\ a_{2,1} & 1 & & O \\ \vdots & \ddots & \ddots & \\ a_{n,1} & \cdots & a_{n,n-1} & 1 \end{bmatrix},$$

it can be shown that for $k \geq 1$, the kth differences of kth row of the Krylov matrix $K(A,\mathbf{v}) = [m_{ij}]$:

$$(\Delta^{k-1}m_{k,1}, \Delta^{k-1}m_{k,2}, \ldots, \Delta^{k-1}m_{k,n-k+1})$$

is the constant sequence $(c_1, \ldots, c_k)$ with $c_1 = v_1$, $c_k = v_1 a_{21} \cdots a_{k,k-1}$ for $k \geq 2$ where $\Delta^k m_{i,j} = \Delta^{k-1}m_{i,j+1} - \Delta^{k-1}m_{i,j}$ with $\Delta^0 m_{i,j} = m_{i,j}$. Further, the matrix $K(A,\mathbf{v})$ can be factored into the product LU where L is a lower triangular matrix with diagonal entries $c_1, c_2, \ldots, c_n$ and $U = P^T$ is the transpose of the $n \times n$ Pascal matrix P whose (i,j)-entry is $\binom{i-1}{j-1}$. A similar observation has been used in [2]. It also follows that

$$\det K(A,\mathbf{v}) = v_1^n \cdot a_{2,1}^{n-1} \cdot a_{3,2}^{n-2} \cdot \cdots \cdot a_{n,n-1}^1. \tag{2}$$

In particular, if A is an infinite triangular matrix then the Krylov matrix $K(A,\mathbf{v})$ is infinite and it is well-defined.

An infinite lower triangular matrix $L = [\ell_{ij}]_{i,j\geq 0}$ is called a *Riordan array* or *Riordan matrix* generated by g and f if there exists a pair of formal power series (g,f), with $f(0) = 0$ over the complex field such that $gf^j = \sum_{i\geq 0} \ell_{ij} z^i$ for $j \geq 0$. Usually we write $L = \mathcal{R}(g,f)$ and it is written in terms of its column generating functions as follows:

$$\mathcal{R}(g,f) = \begin{bmatrix} g & gf & gf^2 & \cdots \end{bmatrix}. \tag{3}$$

The purpose of this paper is to use a new angle to interpret Riordan arrays as Krylov matrices based on a simple observation of (1) and (3). We then provide a new insight that should be useful for an extension of the group of Riordan arrays called the Riordan group. More specifically, in Section 2 we first show that every Riordan array can be expressed as a Krylov matrix. We use this idea in Section 3 to obtain extended Riordan groups. Finally, in Section 4 we study Lie algebras for the extended Riordan groups.