Linear Algebra and its Applications 554 (2018) 371–391

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Periodicity of Grover walks on generalized Bethe trees

Sho Kubota [a], Etsuo Segawa [a], Tetsuji Taniguchi [b], Yusuke Yoshie [a,*]

[a] *Graduate School of Information Sciences, Tohoku University, Aoba, Sendai 980-8579, Japan*
[b] *Department of Electronics and Computer Engineering, Hiroshima Institute of Technology, Hiroshima 731-5193, Japan*

ARTICLE INFO

Article history:
Received 23 January 2018
Accepted 25 May 2018
Available online 29 May 2018
Submitted by R. Brualdi

MSC:
05C05
05C50
05C81
81Q99

Keywords:
Grover walk
Periodicity
Generalized Bethe trees

ABSTRACT

We focus on the periodicity of the Grover walk on the generalized Bethe tree, which is a rooted tree such that in each level the vertices have the same degree. Since the Grover walk is induced by the underlying graph, its properties depend on the graph. In this paper, we say that *the graph induces periodic Grover walks* if and only if there exists $k \in \mathbb{N}$ such that the k-th power of the time evolution operator becomes the identity operator. Our aim is to characterize such graphs. We give the perfect characterizations of the generalized Bethe trees which induce periodic Grover walks.

* Corresponding author.
E-mail address: yusuke.yoshie.r1@dc.tohoku.ac.jp (Y. Yoshie).

https://doi.org/10.1016/j.laa.2018.05.023
0024-3795/

1. Introduction

1.1. Introduction and related works

Quantum walks are introduced as quantum versions of random walks [7]. The quantum walk is induced by a based graph, and the motion of the walker can be regarded as a discrete analogue of scattering of plane wave on the graph [4], [10]. The state of the walker at each time is represented by an ℓ^2-function on an induced Hilbert space, and its evolution is given by a unitary operator on the Hilbert space. Reviews on quantum walks from the viewpoints of several research fields can be seen in the following books, e.g. [16], [17], [19]. Studies of quantum walks have been developed for last decade, and they have been applied to various fields [3], [23]. In particular, the quantum search algorithm is one of the most attractive applications of quantum walks. They enable us to find marked elements on a graph faster than classical random walks [12], [24]. Also, a perfect state transfer from the initial point to the target point can be regarded as a quantum search [6], [13] [22]. Here, we treat the periodicity of a quantum walk. The periodicity appears if there exists an integer k such that the k-th power of the time evolution operator becomes the identity. It also implies that arbitrary state at time k returns to the initial state. So the periodicity can be viewed as a perfect state transfer on the same point and our ultimate purpose is to characterize the graphs in which such a kind of perfect state transfer occurs. If an underlying graph induces periodic quantum walks, then the behavior of the walker is periodic with some periods, and the sequence of the distribution is also periodic. In this paper, we focus on the periodicity of the Grover walk. The Grover walk is a kind of quantum walk, which is uniquely determined by the underlying graph and it is also related to several study fields, e.g. not only quantum searches but also an analysis of the zeta function [14], an isomorphic problem for two cospectral strongly regular graphs [8]. Indeed, the spectrum of the evolution operator of the Grover walk derives from that of the isotropic random walk on the underlying graph. So the spectral analysis of the random walk can be applied to deal the periodicity of the Grover walk.

In this paper, we say that *the graph induces a k-periodic Grover walk* if and only if the k-th power of the time evolution operator becomes the identity first ($k \in \mathbb{N}$). In [9], characterizations of some fixed finite graphs to induce a periodic Grover walk are introduced, and the results are as follows:

- (Complete graphs) The Grover walk on a complete graph K_n is periodic if and only if $n = 2$, or 3, whose periods are $2, 3$, respectively.
- (Complete bipartite graphs) The Grover walk on a complete bipartite graph $K_{r,s}$ is periodic for any $r, s \in \mathbb{N}$ with $r + s \geq 3$, whose period is 4.
- (Strongly regular graphs) The Grover walk on a strongly regular graph SRG(n, k, λ, μ) is periodic if and only if $(n, k, \lambda, \mu) = (5, 2, 0, 1), (2k, k, 0, k)$, or $(3\lambda, 2\lambda, \lambda, 2\lambda)$, whose periods are $5, 4, 12$, respectively. These graphs are nothing but $C_5, K_{k,k}$, and $K_{\lambda,\lambda,\lambda}$, respectively.