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Mixed forward–backward stability of the two-level orthogonal Arnoldi method for quadratic problems ☆

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We revisit the numerical stability of the two-level orthogonal Arnoldi (TOAR) method for computing an orthonormal basis of a second-order Krylov subspace associated with two given matrices. We show that the computed basis is close to a basis for a second-order Krylov subspace associated with nearby matrices, provided that the norms of the given matrices are not too large or too small. Thus, we provide conditions that guarantee the numerical stability of the TOAR method in computing orthonormal bases of second-order Krylov subspaces. We also study scaling the quadratic problem for improving the numerical stability of the TOAR procedure when the norms of the matrices are too large or too small.

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1. Introduction

Given two complex matrices $A, B \in \mathbb{C}^{n\times n}$ and two starting vectors $r_{-1}, r_0 \in \mathbb{C}^n$, if we define the sequence $r_{-1}, r_0, r_1, \ldots, r_{k-1}$ by the recurrence relation

$$r_i = Ar_{i-1} + Br_{i-2}, \quad \text{for } i = 1, 2, \ldots, k-1,$$

then, the *second-order Krylov subspace* associated with A and B, introduced by Bai and Su [2], is the subspace

$$\mathcal{G}_k(A, B; r_{-1}, r_0) := \operatorname{span}\{r_{-1}, r_0, r_1, \ldots, r_{k-1}\}.$$

Projection methods based on second-order Krylov subspaces have been found to be reliable procedures for obtaining good approximations to the solutions of (structured) quadratic eigenvalue problems [2,11], and for model order reduction of second-order dynamical systems [1,11] and second-order time-delay systems [21]. These procedures start by computing an orthonormal set of vectors $\{q_1, q_2, \ldots, q_{k+1}\}$ such that

$$\operatorname{span}\{q_1, q_2, \ldots, q_{k+1}\} = \mathcal{G}_k(A, B; r_{-1}, r_0).$$

They continue by projecting the problem onto the subspace $\mathcal{G}_k(A, B; r_{-1}, r_0)$, reducing the size of the original problem. Finally, the projected problem is solved by using standard algorithms for small/medium-sized dense matrices. The convergence of these projection methods for quadratic eigenvalue problems is studied in [9].

The *second-order Arnoldi (SOAR)* method and the *two-level orthogonal Arnoldi (TOAR)* method [2,11,18], are two well-known algorithms for computing orthonormal bases of second-order Krylov subspaces. Both methods compute such bases by embedding the second-order Krylov subspaces into standard Krylov subspaces. Moreover, while the SOAR method is prone to numerical instability [11], the analysis performed in [11] provides solid theoretical evidence of the numerical stability of the TOAR method. More precisely, the TOAR method is backward stable in computing an orthonormal basis of the Krylov subspace in which the second-order Krylov subspace is embedded. In this work, we extend this result by showing that the computed orthonormal basis for the second-order Krylov subspace is close (in the standard subspace metric sense [16]) to a second-order Krylov subspace associated with matrices $A + \Delta A$ and $B + \Delta B$. This result is stated in Corollary 3.3, which is a consequence of the more general Theorem 3.1. These two results are the major contributions of this work. Additionally, in Section 4, we study how scaling the original quadratic problem affects the norms of ΔA and ΔB.

The notation used in the rest of the paper is as follows. We use lowercase letters for vectors and uppercase letters for matrices. In addition, we use boldface letters to indicate that a matrix (resp. vector) will be considered as a 2×1 block-matrix (resp. block-vector), whose blocks are denoted with superscripts as in