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Maximal determinants of combinatorial matrices

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We prove that $\det A \leq 6^{\frac{n}{6}}$ whenever $A \in \{0,1\}^{n\times n}$ contains at most $2n$ ones. We also prove an upper bound on the determinant of matrices with the k-consecutive ones property, a generalisation of the consecutive ones property, where each row is allowed to have up to k blocks of ones. Finally, we prove an upper bound on the determinant of a path-edge incidence matrix in a tree and use that to bound the leaf rank of a graph in terms of its order.

1. Introduction

There is a rich tradition of bounding the determinants of matrices with all entries 0 or 1 (or −1 and 1). Yet, even very simple questions remain open. For instance:

Problem 1. Given a matrix $A \in \{0,1\}^{n\times n}$ with at most $2n$ ones, how large can its determinant be?

In fact, Problem 1 is open for any linear number of ones in A, and we consider the case of at most $2n$ ones as a simple and restricted representative. Furthermore, this case

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naturally relates to edge-vertex incidence matrices of graphs, which we exploit below. We give an answer to this question as well as to questions about the determinants of similarly simple matrices.

The most prominent question in this area is probably *Hadamard's maximal determinant problem*: Given $A \in \{-1,1\}^{n\times n}$, how large can $\det A$ be? A partial answer lies in *Hadamard's inequality* [10]. For this, let us denote the ith row of a matrix $A \in \mathbb{R}^{n\times n}$ by $A_{i,\bullet}$, and similarly we write $A_{\bullet,j}$ for the jth column. Then

$$|\det(A)| \le \prod_{i=1}^{n} ||A_{i,\bullet}||_2. \tag{1}$$

If A is a $-1/1$-matrix, then (1) implies $|\det(A)| \le n^{n/2}$, and A is a *Hadamard matrix* of order n if this inequality holds with equality. It is known that a Hadamard matrix of order n can exist only if $n \in \{1,2\} \cup 4\mathbb{N}$. Paley's conjecture [15] states that for all these orders, Hadamard matrices actually do exist.

Coming back to Problem 1, with the inequality of the arithmetic and geometric mean, it is not hard to derive from (1) that $\det A \le 2^{\frac{n}{2}}$ for any $A \in \{0,1\}^{n\times n}$ with at most $2n$ ones. Ryser improved this estimate, and at the same time extended it to cover more general numbers of 1-entries in A:

Theorem 2 *(Ryser [16]). Let $A \in \{0,1\}^{n\times n}$ be a matrix containing exactly kn ones. If $n \ge 2$ and $1 \le k \le \frac{n+1}{2}$ then, for $\lambda = k(k-1)/(n-1)$, it follows that*

$$|\det A| \le k(k-\lambda)^{\frac{1}{2}(n-1)}.$$

Moreover, Ryser showed that the bound is tight for many pairs of k, n; see next section. Inspecting Ryser's bound, we see that in the situation of Question 1, when $k \le 2$, it yields a bound of $\det A \le 2(2-o(1))^{\frac{n-1}{2}}$, which is not much better than the bound of $2^{\frac{n}{2}}$ that we get from Hadamard's inequality. Given that Ryser's bound is tight for many k, n, does that mean that we cannot hope for a better bound? No, it turns out. Our main result is:

Theorem 3. *If $A \in \{0,1\}^{n\times n}$ has at most $2n$ non-zero entries, then $|\det(A)| \le 2^{n/6} \cdot 3^{n/6} \approx 1.348^n$.*

While a substantial improvement on the bound of $2^{\frac{n}{2}} \approx 1.414^n$, the bound in Theorem 3 is almost certainly not best possible. Indeed, the best lower bound we have found comes from matrices of the kind

$$A = \operatorname{diag}(C, \dots, C), \text{ where } C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

that have determinant $2^{\frac{n}{3}} \approx 1.260^n$. We believe that larger determinants are impossible: