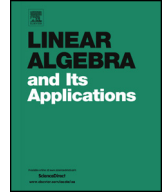




Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa



Log-majorizations for the (symplectic) eigenvalues of the Cartan barycenter



Fumio Hiai^a, Yongdo Lim^{b,*}

^a *Tohoku University¹, Hakusan 3-8-16-303, Abiko 270-1154, Japan*

^b *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea*

ARTICLE INFO

Article history:

Received 7 February 2018

Accepted 23 April 2018

Available online 4 May 2018

Submitted by P. Semrl

MSC:

15A42

47A64

47B65

47L07

Keywords:

Positive definite matrix

Riemannian trace metric

Cartan barycenter

Probability measure

(Symplectic) eigenvalue

Log-majorization

ABSTRACT

In this paper we show that the eigenvalue map and the symplectic eigenvalue map of positive definite matrices are Lipschitz for the Cartan–Hadamard Riemannian metric, and establish log-majorizations for the (symplectic) eigenvalues of the Cartan barycenter of integrable probability Borel measures. This leads a version of Jensen's inequality for geometric integrals of matrix-valued integrable random variables.

© 2018 Elsevier Inc. All rights reserved.

* Corresponding author.

E-mail addresses: fumio.hiai@gmail.com (F. Hiai), yelim@skku.edu (Y. Lim).

¹ Emeritus.

1. Introduction

Let $\mathbb{S}_n$ be the Euclidean space of $n \times n$ real symmetric matrices equipped with the trace inner product $\langle X, Y \rangle = \operatorname{tr}(XY)$. Let $\mathbb{P}_n \subset \mathbb{S}_n$ be the open convex cone of real positive definite matrices, which is a smooth Riemannian manifold with the Riemannian trace metric $\langle X, Y \rangle_A = \operatorname{tr} A^{-1} X A^{-1} Y$, where $A \in \mathbb{P}_n$ and $X, Y \in \mathbb{S}_n$. This is an important example of Cartan–Hadamard manifolds, simply connected complete Riemannian manifolds with non-positive sectional curvature (the canonical 2-tensor is non-negative). The Riemannian distance between $A, B \in \mathbb{P}_n$ with respect to the above metric is given by $\delta(A, B) = \|\log A^{-1/2} B A^{-1/2}\|_2$, where $\|X\|_2 = (\operatorname{tr} X^2)^{1/2}$ for $X \in \mathbb{S}_n$.

One of recent active research topics on this Riemannian manifold $\mathbb{P}_n$ is the Cartan mean (alternatively the Riemannian mean, the Karcher mean)

$$G(A_1, \dots, A_m) := \arg \min_{X \in \mathbb{P}} \sum_{j=1}^m \delta^2(A_j, X),$$

where the minimizer exists uniquely. This is a multivariate extension of the two-variable geometric mean $A \# B := A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$, which is the unique midpoint between A and B for the Riemannian trace metric, and it retains most of its attractive properties; for instance, joint homogeneity, monotonicity, joint concavity, and the arithmetic–geometric–harmonic mean inequalities. It also extends the multivariate geometric mean on $\mathbb{R}_+^n \subset \mathbb{P}_n$, where $\mathbb{R}_+ = (0, \infty)$, via the embedding into diagonal matrices, $(a_1, \dots, a_n) \mapsto \operatorname{diag}(a_1, \dots, a_n)$.

The Cartan mean extends uniquely to a contractive (with respect to the Wasserstein metric) barycentric map on the Wasserstein space of L^1 -probability measures;

$$G : \mathcal{P}^1(\mathbb{P}_n) \rightarrow \mathbb{P}_n,$$

where a probability Borel measure μ belongs to $\mathcal{P}^1(\mathbb{P}_n)$ if $\int_{\mathbb{P}_n} \delta(A, X) d\mu(A) < \infty$ for some $X \in \mathbb{P}_n$. The Cartan barycenter plays a fundamental role in the theory of integrations (random variables, expectations and variances). Let $(\Omega, \mathbf{P})$ be a probability space and let $L^1(\Omega; \mathbb{P}_n)$ be the space of measurable functions $\varphi : \Omega \rightarrow \mathbb{P}_n$ such that $\int_{\Omega} \delta(\varphi(\omega), X) d\mathbf{P}(\omega) < \infty$ for some $X \in \mathbb{P}_n$. Then the “geometric” integral of $\varphi \in L^1(\Omega; \mathbb{P}_n)$ is naturally defined as

$$\int_{\Omega}^{(G)} \varphi(\omega) d\mathbf{P}(\omega) := G(\varphi_* \mathbf{P}).$$

Here, we use the notation $\int_{\Omega}^{(G)}$ to avoid the confusion with the usual $\int_{\Omega}$ in the Euclidean (or arithmetic) sense, that is, $\int_{\Omega} \varphi(\omega) d\mathbf{P}(\omega) = \mathcal{A}(\varphi_* \mathbf{P})$, where $\mathcal{A} : \mathcal{P}^{\infty}(\mathbb{P}_n) \rightarrow \mathbb{P}_n$ is the arithmetic barycenter on the space of bounded probability measures and $\varphi_* \mathbf{P}$ is the push-forward measure by φ , that is, $\varphi_* \mathbf{P}(B) = \mathbf{P}(\varphi^{-1}(B))$ for any Borel set B in $\mathbb{P}_n$.

Download English Version:

<https://daneshyari.com/en/article/8897773>

Download Persian Version:

<https://daneshyari.com/article/8897773>

[Daneshyari.com](https://daneshyari.com)