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Log-majorizations for the (symplectic) eigenvalues of the Cartan barycenter

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ABSTRACT

In this paper we show that the eigenvalue map and the symplectic eigenvalue map of positive definite matrices are Lipschitz for the Cartan–Hadamard Riemannian metric, and establish log-majorizations for the (symplectic) eigenvalues of the Cartan barycenter of integrable probability Borel measures. This leads a version of Jensen's inequality for geometric integrals of matrix-valued integrable random variables.

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1. Introduction

Let $\mathbb{S}_n$ be the Euclidean space of $n \times n$ real symmetric matrices equipped with the trace inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$. Let $\mathbb{P}_n \subset \mathbb{S}_n$ be the open convex cone of real positive definite matrices, which is a smooth Riemannian manifold with the Riemannian trace metric $\langle X, Y\rangle_A = \mathrm{tr}\, A^{-1}XA^{-1}Y$, where $A \in \mathbb{P}_n$ and $X, Y \in \mathbb{S}_n$. This is an important example of Cartan–Hadamard manifolds, simply connected complete Riemannian manifolds with non-positive sectional curvature (the canonical 2-tensor is non-negative). The Riemannian distance between $A, B \in \mathbb{P}_n$ with respect to the above metric is given by $\delta(A, B) = \|\log A^{-1/2}BA^{-1/2}\|_2$, where $\|X\|_2 = (\mathrm{tr}\, X^2)^{1/2}$ for $X \in \mathbb{S}_n$.

One of recent active research topics on this Riemannian manifold $\mathbb{P}_n$ is the Cartan mean (alternatively the Riemannian mean, the Karcher mean)

$$G(A_1, \dots, A_m) := \underset{X\in\mathbb{P}}{\arg\min} \sum_{j=1}^{m} \delta^2(A_j, X),$$

where the minimizer exists uniquely. This is a multivariate extension of the two-variable geometric mean $A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, which is the unique midpoint between A and B for the Riemannian trace metric, and it retains most of its attractive properties; for instance, joint homogeneity, monotonicity, joint concavity, and the arithmetic–geometric–harmonic mean inequalities. It also extends the multivariate geometric mean on $\mathbb{R}_+^n \subset \mathbb{P}_n$, where $\mathbb{R}_+ = (0, \infty)$, via the embedding into diagonal matrices, $(a_1, \dots, a_n) \mapsto \mathrm{diag}(a_1, \dots, a_n)$.

The Cartan mean extends uniquely to a contractive (with respect to the Wasserstein metric) barycentric map on the Wasserstein space of L^1-probability measures;

$$G : \mathcal{P}^1(\mathbb{P}_n) \to \mathbb{P}_n,$$

where a probability Borel measure μ belongs to $\mathcal{P}^1(\mathbb{P}_n)$ if $\int_{\mathbb{P}_n} \delta(A, X)\, d\mu(A) < \infty$ for some $X \in \mathbb{P}_n$. The Cartan barycenter plays a fundamental role in the theory of integrations (random variables, expectations and variances). Let $(\Omega, \mathbf{P})$ be a probability space and let $L^1(\Omega; \mathbb{P}_n)$ be the space of measurable functions $\varphi : \Omega \to \mathbb{P}_n$ such that $\int_\Omega \delta(\varphi(\omega), X)\, d\mathbf{P}(\omega) < \infty$ for some $X \in \mathbb{P}_n$. Then the "geometric" integral of $\varphi \in L^1(\Omega; \mathbb{P}_n)$ is naturally defined as

$$\int_\Omega^{(G)} \varphi(\omega)\, d\mathbf{P}(\omega) := G(\varphi_*\mathbf{P}).$$

Here, we use the notation $\int_\Omega^{(G)}$ to avoid the confusion with the usual $\int_\Omega$ in the Euclidean (or arithmetic) sense, that is, $\int_\Omega \varphi(\omega)\, d\mathbf{P}(\omega) = \mathcal{A}(\varphi_*\mathbf{P})$, where $\mathcal{A} : \mathcal{P}^\infty(\mathbb{P}_n) \to \mathbb{P}_n$ is the arithmetic barycenter on the space of bounded probability measures and $\varphi_*\mathbf{P}$ is the push-forward measure by φ, that is, $\varphi_*\mathbf{P}(B) = \mathbf{P}(\varphi^{-1}(B))$ for any Borel set B in $\mathbb{P}_n$.