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# On the distance spectra of threshold graphs ☆

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A B S T R A C T

A graph is called a threshold graph if it does not contain induced $C_4$, $P_4$ or $2K_2$. Such graphs have numerous applications in computer science and psychology, and they also have nice spectral properties. In this paper, we consider the distance matrix of a connected threshold graph. We show that there are no distance eigenvalues of threshold graphs lying in the interval $(-2, -1)$ and all the eigenvalues, other than $-2$ or $-1$, are simple. Besides, we determine all threshold graphs with distinct distance eigenvalues.



## 1. Introduction

Let $\Gamma$ be a connected graph with vertex set $V = \{v_1, v_2 \ldots, v_n\}$. The *distance* between $v_i$ and $v_j$, denoted by $d(v_i, v_j)$ (or $d_{i,j}$ for short), is the length of a shortest path between $v_i$ and $v_j$. The *neighbourhood* of $v_i$ is the collection of all vertices adjacent to $v_i$, denoted by $N(v_i)$, that is, $N(v_i) = \{v_j \mid d(v_i, v_j) = 1\}$. The *diameter* of $\Gamma$, denoted by $d(\Gamma)$, is the

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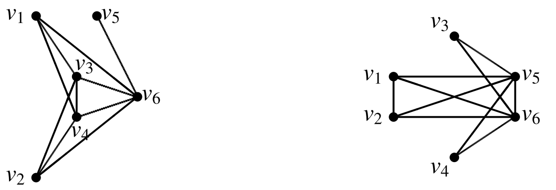


$\Gamma_1$: $(0^2, 1^2, 0, 1)$ $\qquad$ $\Gamma_2$ : $(0, 1, 0^2, 1^2)$

$\mathrm{Spec}_D(\Gamma_1) = \{6.86, -0.45, -0.73, -1, -2, -2.69\}$ $\qquad$ $\mathrm{Spec}_D(\Gamma_2) = \{6.89, -0.34, -1, -1, -2, -2.54\}$

**Fig. 1.** Example of threshold graphs.

largest distance in $\Gamma$. The *distance matrix* of $\Gamma$, denoted by $\mathcal{D}(\Gamma)$, is the $n \times n$ matrix whose $(i, j)$-entry is equal to $d(v_i, v_j)$, for $1 \leq i, j \leq n$. The eigenvalues of $\mathcal{D}(\Gamma)$, listed by $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$, are the *distance eigenvalues* of $\Gamma$. The multiset of distance eigenvalues of $\Gamma$ is the *distance spectrum* of $\Gamma$, always denoted by $\mathrm{Spec}_D(\Gamma) = \{[\partial_1]^{m_1}, \ldots, [\partial_s]^{m_s}\}$, where $\partial_1 > \cdots > \partial_s$ and the superscript $m_i$ is the multiplicity of $\partial_i$. We refer the reader to the survey paper [1], for more details about the backgrounds and applications of distance matrix.

If a graph contains no induced $C_4$, $P_4$ or $2K_2$, then it is a *threshold graph*. A threshold graph can be obtained by repeatedly performing one of the following two operations: (a) adding a new vertex adjacent to none of the former vertices (such vertex is called a separate vertex); (b) adding a new vertex adjacent to all of the former vertices (such vertex is called a dominating vertex). Let $\Gamma$ be a threshold graph of order $n$, whose vertex set is labelled as $\{v_1, \ldots, v_n\}$ such that $v_i$ is the added vertex in the $i$-th step of the operations. We can use a $\{0, 1\}$-sequence $b = (b_1, \ldots, b_n)$ to represent $\Gamma$, where $b_i = 0$ if $v_i$ is a separate vertex and $b_i = 1$ if $v_i$ is a dominating vertex. As usual, we set $b_1 = 0$. Obviously, $\Gamma$ is connected if and only if $b_n = 1$. Note that the successive separate vertices have the same properties and the successive dominating vertices have the same properties. We collect the successive 0s and 1s together in $b$. Therefore, the sequence $b$ can be written as $b = (0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$, where $s_i, t_i \geq 1$ for $1 \leq i \leq m$ and $m \geq 1$. This sequence is the *representation sequence* of $\Gamma$, and $\Gamma$ is uniquely determined by its representation sequence. For example, the threshold graphs with representation sequences $(0^2, 1^2, 0, 1)$ and $(0, 1, 0^2, 1^2)$ are shown in Fig. 1.

Threshold graphs were first introduced by Chvátal and Hammer [19] and Henderson and Zalcstein [8] in 1977. After that, threshold graphs have been paid close extensive attention because of their numerous applications in computer science and psychology [19]. Recently, many mathematicians studied the eigenvalues of the adjacency matrix of threshold graphs. In 2011, Sciriha and Farrugia [21] gave some spectral properties of adjacency eigenvalues of threshold graphs. In 2013, Bapat [2] obtained the determinant of the adjacency matrix of threshold graphs and he gave the nullity of threshold graphs as well. In the same year, Jacobs et al. [12] presented an $O(n)$ algorithm for constructing a diagonal matrix congruent to $B_x = A + xI$ for any $x$. By using this method, they published several papers [12,14,15] to investigate the properties of adjacency eigenvalues