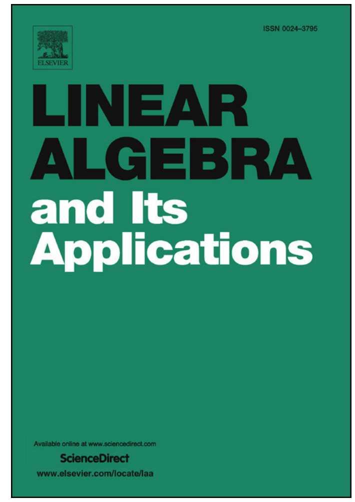

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SOME RESULTS ON STRONGLY OPERATOR CONVEX FUNCTIONS AND OPERATOR MONOTONE FUNCTIONS

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Abstract. This paper concerns three classes of real-valued functions on intervals, operator monotone functions, operator convex functions, and strongly operator convex functions. Strongly operator convex functions were previously treated in [3] and [4], where operator algebraic semicontinuity theory or operator theory were substantially used. In this paper we provide an alternate treatment that uses only operator inequalities (or even just matrix inequalities). We show also that if t_0 is a point in the domain of a continuous function f, then f is operator monotone if and only if $(f(t)-f(t_0))/(t-t_0)$ is strongly operator convex. Using this and previously known results, we provide some methods for constructing new functions in one of the three classes from old ones. We also include some discussion of completely monotone functions in this context and some results on the operator convexity or strong operator convexity of $\varphi\circ f$ when f is operator convex or strongly operator convex.

1. Introduction

Let $f(t)$ be a real continuous function defined on a non-degenerate interval J in the real axis. For a bounded self-adjoint operator (or matrix) A on a Hilbert space $\mathbf{H}$ whose spectrum is in J, $f(A)$ is well-defined. Then f is called an *operator monotone function* on J, denoted by $f\in\mathbf{P}(J)$, if $f(A)\leqq f(B)$, whenever $A\leqq B$. We call f *operator decreasing* if $-f$ is operator monotone. The Löwner theorem [9] says that a C^1-function f is operator monotone on an open interval J if and only if the Löwner kernel function $K_f(t,s)$ defined by

$$K_f(t,s)=\frac{f(t)-f(s)}{t-s}\quad (t\neq s),\quad K_f(t,t)=f'(t),$$

is positive semi-definite on J, and that such a function f possesses a holomorphic extension $f(z)$ into the open upper half plane Π_+ which maps Π_+ into itself (unless f is constant), namely $f(z)$ is a *Pick function*. Since $f(z)$ also has a holomorphic extension to the open lower half plane Π_-, then $f(t)$ has a holomorphic extension to $J\cup\Pi_+\cup\Pi_-$. In this case, it follows from Herglotz's theorem that $f(t)$ has an integral representation:

$$f(t)=\alpha+\beta t+\int_{-\infty}^{\infty}\left(\frac{1}{x-t}-\frac{1}{x-t_0}\right)d\nu(x), \tag{1}$$

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