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Spectral characterizations of signed cycles

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ABSTRACT

A signed graph is a pair like (G, σ), where G is the underlying graph and $\sigma : E(G) \to \{-1, +1\}$ is a sign function on the edges of G. In this paper we study the spectral determination problem for signed n-cycles (C_n, σ) with respect to the adjacency spectrum and the Laplacian spectrum. In particular, for the Laplacian spectrum, we prove that balanced odd cycles and unbalanced cycles, denoted, respectively, by C^+_{2n+1} and C^-_n, are uniquely determined by their Laplacian spectra (i.e., they are DLS). On the other hand, we determine all Laplacian cospectral mates of the balanced even cycles C^+_{2n}, so that we show that C^+_{2n} is not DLS. The same problem is then considered for the adjacency spectrum, hence we prove that odd signed cycles, namely, C^+_{2n+1} and C^-_{2n+1}, are uniquely determined by their (adjacency) spectrum (i.e., they are DS). Moreover, we find cospectral mates for the even signed cycles C^+_{2n} and C^-_{2n}, and we show that, except the signed cycle C^-_4, even signed cycles are not DS and we provide almost all cospectral mates.

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1. Introduction and terminology

By a graph $G = (V, E)$, we mean a simple graph without loops or multiple edges, where $V = V(G)$ and $E = E(G)$ are the set of vertices and edges of G, respectively. Let $\sigma : E(G) \to \{-1, +1\}$ be a mapping defined on the edges of G. Then, the pair $\Gamma = (G, \sigma)$ is called a *signed graph*, where G is its underlying graph. We also say that G is an *unsigned* graph. If $v \in V$ is a vertex of Γ, then by $d_\Gamma(v)$ we mean the degree of v in G. A cycle in Γ is said to be *positive* if it contains an even number of negative edges, otherwise the cycle is called *negative*. A signed graph is said to be *balanced* if all its cycles are positive, otherwise it is *unbalanced*. Most of the usual graph theory glossary directly extends to signed graphs; for basic notation on signed graphs the reader is referred to [15]. Signed graphs were introduced by Harary [10] in connection with the study of the theory of social psychology. Among the fundamental works in this area, we can refer to Zaslavsky's papers [17] and [18]. Also, for a possibly complete bibliography on signed graphs see [14].

Now, let $\theta : V \to \{-1, +1\}$ be a sign function. The *switching* Γ by θ is a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$, where $\sigma^\theta(e) = \theta(v_i)\sigma(e)\theta(v_j)$ for each edge $e = v_iv_j \in E(G)$. Two signed graphs Γ and Λ are *switching equivalent* and we write $\Gamma \sim \Lambda$ if there exists a switching function θ such that $\Lambda = \Gamma^\theta$. Obviously, $\sim$ is an equivalence relation on signed graphs with the same underlying graph. It is worthy to notice that the signature switching does not affect the sign of the cycles, i.e., Γ and Γ^θ share the set of positive cycles. Additionally, the sign of non-cyclic edges (say, bridges) is irrelevant. Therefore, if G is a tree, then all signed graphs on G are switching equivalent. Also, for unicyclic graphs, there are exactly two different switching equivalent classes. In particular, C_n, the cycle of order n, has two switching equivalent classes; the positive (or, balanced) cycles and the negative (or, unbalanced) cycles which are denoted by C_n^+ and C_n^-, respectively.

Similarly to unsigned graphs, matrices can be used to study signed graphs. Let $\{v_1, \ldots, v_n\}$ be the vertices of Γ. Then, the *adjacency matrix* of Γ, $A(\Gamma)$ or simply A, is defined as following: $[A]_{ij} = \sigma(v_i, v_j)a_{ij}$, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The polynomial $\phi(\Gamma, \lambda) = \det(\lambda I - A) = \lambda^n + a_1^\sigma \lambda^{n-1} + \cdots + a_{n-1}^\sigma \lambda + a_n^\sigma$ is called the *characteristic polynomial* of Γ and its roots are *eigenvalues* or *spectrum* of Γ, denoted by $\mathrm{Spec}(\Gamma)$. Note that the characteristic polynomial of the null graph, the graph with no vertices and edges, is defined to be the constant polynomial 1.

Also, the matrix $L(\Gamma) = D(\Gamma) - A(\Gamma)$, or simply L, is called the *Laplacian matrix* of Γ, where $D(\Gamma)$ is the $n \times n$ diagonal matrix with $d(v_1), \ldots, d(v_n)$ as the diagonal entries (and all other entries 0). The polynomial $\psi(\Gamma, \mu) = \det(\mu I - L)$ is called the *Laplacian polynomial* of Γ and its roots are *Laplacian eigenvalues* or *Laplacian spectrum* of Γ, denoted by $\mathrm{Spec}_L(\Gamma)$. Observe that permuting the vertices in the underlying graph does not change the spectrum or the L-spectrum; a similar feature holds with signature switching: Γ and Γ^θ share the same adjacency or Laplacian spectrum. Therefore, if a signed graph can be switched into an isomorphic copy of another signed graph, then the two signed graphs are said to be *switching isomorphic* and they will get the same