Linear Algebra and its Applications 553 (2018) 328–361

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Lie product and local spectrum preservers

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A R T I C L E I N F O

Article history:
Received 19 February 2018
Accepted 9 May 2018
Available online 17 May 2018
Submitted by P. Semrl

MSC:
primary 47B49
secondary 47A10, 47A11

Keywords:
Nonlinear preservers
Local spectrum
The single-valued extension property
Lie product

A B S T R A C T

Let X and Y be two infinite-dimensional complex Banach spaces, and fix two nonzero vectors $x_0 \in X$ and $y_0 \in Y$. Let $\mathcal{B}(X)$ (resp. $\mathcal{B}(Y)$) denote the algebra of all bounded linear operators on X (resp. on Y), and let $\mathcal{E}_{x_0}(X)$ be the collection of all operators $T \in \mathcal{B}(X)$ for which x_0 is an eigenvector and $(T - r\mathbf{1}_X)^2$ is a nonzero scalar operator for some scalar $r \in \mathbb{C}$. We show that a map φ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfies

$$\sigma_{\varphi(T)\varphi(S)-\varphi(S)\varphi(T)}(y_0) = \sigma_{TS-ST}(x_0), \ (T,\ S \in \mathcal{B}(X)),$$

if and only if there are two functions $\eta : \mathcal{B}(X) \to \mathbb{C}$ and $\xi : \mathcal{B}(X) \to \{-1,\ 1\}$, and a bijective bounded linear mapping $A : X \to Y$ such that $Ax_0 = y_0$, the function ξ is constant on $\mathcal{B}(X)\backslash\mathcal{E}_{x_0}(X)$, and

$$\varphi(T) = \xi(T)ATA^{-1} + \eta(T)\mathbf{1}_Y$$

for all $T \in \mathcal{B}(X)$.

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https://doi.org/10.1016/j.laa.2018.05.013
0024-3795/

1. Introduction and statement of main result

Throughout this paper, X and Y denote infinite-dimensional complex Banach spaces, and let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. Let $\mathcal{B}(X)$ (resp. $\mathcal{B}(Y)$) denote the algebra of all bounded linear operators on X (resp. on Y), and denote its identity operator by $\mathbf{1}$. The *local resolvent set*, $\rho_T(x)$, of an operator $T \in \mathcal{B}(X)$ at a point $x \in X$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\zeta : U \to X$ such that $(T-\lambda)\zeta(\lambda) = x$, $(\lambda \in U)$. The *local spectrum* of T at x is $\sigma_T(x) := \mathbb{C}\backslash\rho_T(x)$, and is obviously a closed subset of $\sigma(T)$, the spectrum of T. An operator $T \in \mathcal{B}(X)$ is said to have the *single-valued extension property* (SVEP) if for every open subset U of $\mathbb{C}$, the equation $(T-\lambda)\varphi(\lambda) = 0$, $(\lambda \in U)$, has no nontrivial X-valued analytic solution φ. Recall that $\sigma_T(x) \neq \emptyset$ for all nonzero vectors x in X precisely when T has the SVEP, and note that every operator $T \in \mathcal{B}(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. Note that if $T \in \mathcal{B}(X)$ has SVEP and $Tx = \alpha x$ for some nonzero vector $x \in X$ and $\alpha \in \mathbb{C}$, then $\sigma_T(x) = \{\alpha\}$. Note also that if $T \in \mathcal{B}(X)$ has SVEP and $Tx = \alpha y$ for some vectors $x,\ y \in X$ and $\alpha \in \mathbb{C}$, then $\sigma_T(y) \subset \sigma_T(x) \subset \sigma_T(y) \cup \{0\}$. Moreover, $\sigma_T(Rx) \subset \sigma_T(x)$ for all commuting operators T and R in $\mathcal{B}(X)$ and all $x \in X$. Furthermore, $\sigma_T(x+y) \subset \sigma_T(x) \cup \sigma_T(y)$ for all $T \in \mathcal{B}(X)$ and $x,\ y \in X$, and the equality holds if $\sigma_T(x) \cap \sigma_T(y) = \emptyset$. For more information on local spectral theory, the interested reader may consult the remarkable books of Aiena [3] and of Laursen and Neumann [18].

Beside the recent works [1,2,5–10] on nonlinear *local spectra preserver problems*, we are motivated by several papers where the main interest is focused on characterizing nonlinear maps preserving a spectral quantity or relation of Jordan and Lie products of operators or matrices; see for instance [11–17,19–21]. In [5, Theorem 2.1], the second and third authors described the form of all maps preserving the local spectrum of Jordan product of operators on a complex Banach space. They showed that a map φ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfies

$$\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0) \;=\; \sigma_{TS+ST}(x_0),\ (T,\ S \in \mathcal{B}(X)) \tag{1.1}$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and $\varphi(T) = \pm ATA^{-1}$ for all $T \in \mathcal{B}(X)$. In the present paper, we characterize all maps φ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ preserving the local spectrum of Lie product $[S,T] := ST - TS$ of operators. Note that, since $[S,T] = [S+\lambda\mathbf{1}, T+\mu\mathbf{1}]$ for all $S,\ T \in \mathcal{B}(X)$ and $\lambda,\ \mu \in \mathbb{C}$, for every function $\eta : \mathcal{B}(X) \to \mathbb{C}$ and a bijective map $A : X \to Y$ such that $Ax_0 = y_0$, the map

$$\varphi(T) = \pm ATA^{-1} + \eta(T)\mathbf{1}_Y,\ (T \in \mathcal{B}(X)) \tag{1.2}$$

satisfies

$$\sigma_{[\varphi(T),\ \varphi(S)]}(y_0) \;=\; \sigma_{[T,\ S]}(x_0),\ (T,\ S \in \mathcal{B}(X)). \tag{1.3}$$