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Generators and relations for the unitary group of a skew hermitian form over a local ring

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ABSTRACT

Let $(S, *)$ be an involutive local ring and let $U(2m, S)$ be the unitary group associated to a nondegenerate skew hermitian form defined on a free S-module of rank $2m$. A presentation of $U(2m, S)$ is given in terms of Bruhat generators and their relations. This presentation is used to construct an explicit Weil representation of the symplectic group $Sp(2m, R)$ when $S = R$ is commutative and $*$ is the identity.

When S is commutative but $*$ is arbitrary with fixed ring R, an elementary proof that the special unitary group $SU(2m, S)$ is generated by unitary transvections is given. This is used to prove that the reduction homomorphisms $SU(2m, S) \to SU(2m, \tilde{S})$ and $U(2m, S) \to U(2m, \tilde{S})$ are surjective for any factor ring $\tilde{S}$ of S. The corresponding results for the symplectic group $Sp(2m, R)$ are obtained as corollaries when $*$ is the identity.

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1. Introduction

We are concerned with the generation, presentation and representations of the unitary group $U(2m, S)$ of rank $2m$ over an involutive local ring $(S, *)$ associated to a standard nondegenerate skew hermitian form.

Presentations of classical groups over fields in terms of elementary matrices can be found in [12, Theorems 2.3.4*, 2.3.6, 2.3.8, 6.5.7, 6.5.8, 6.5.9]. For other types of presentations for these groups see [1,3,7,11]. For classical groups over rings the problem is more difficult. Pantoja [20] finds a presentation for the group $SSL_*(2, A)$ when A is a simple artinian ring in terms of Bruhat generators. Here $SL_*(2, A)$ is a rank 2 $*$-analogue of $SL(2, F)$, F a field, isomorphic to the rank $2m$ unitary group over the division ring underlying A. The presentation given in [20] is stated for the subgroup $SSL_*(2, A)$ of $SL_*(2, A)$ generated by the Bruhat elements and, in general, $SSL_*(2, A)$ is a proper subgroup of $SL_*(2, A)$. We follow [20] and a prior paper by Pantoja and Soto Andrade [18] in order to extend and sharpen the results of [20] by giving a Bruhat presentation of $U(2m, S)$, where $(S, *)$ is an involutive local ring, not necessarily commutative.

As an application of the above presentation we construct an explicit Weil representation of the symplectic group $Sp(2m, R) = U(2m, R)$ when $S = R$ is commutative and $*$ is the identity. We simply assign linear operators to the Bruhat generators and verify that the defining relations are satisfied. We then demonstrate that the representation thus defined is a Weil representation, in the sense that it is formed by intertwining operators for the Schrödinger representation of the Heisenberg group on which $Sp(2m, R)$ acts by means of group automorphisms. We refer the reader to [10,23], where Weil representations of other groups of the form $SL_*(2, A)$ have also been constructed using generators and relations, and these representations were verified to be Weil by a different method, namely by appealing to Howe's theory of reductive dual pairs. See [19] for more on reductive dual pairs, the Weil representation and the theta correspondence.

We also consider the generation of the special unitary group $SU(2m, S)$ by unitary transvections, where $(S, *)$ a is local, commutative involutive ring with fixed ring R. The classical field case can be found in [6]. The ring case, with transvections replaced by elementary Eichler transformations, can be found in [12, Theorem 9.2.6]. Transvections themselves were proven by Baeza [2] to generate the special unitary group associated to a nondegenerate *hermitian* form of hyperbolic rank ≥ 1, provided all of R is the image of the trace map $s \mapsto s + s^*$. Unlike the field case, the skew hermitian case cannot be derived from the hermitian case, since $*$-skew hermitian units need not exist (they certainly do not exist when $(S, *)$ is ramified). We give an elementary proof that $SU(2m, S)$ is generated by unitary transvections and use this to prove that both reduction homomorphisms $SU(2m, S) \to SU(2m, \tilde{S})$ and $U(2m, S) \to U(2m, \tilde{S})$ are surjective for any factor ring $\tilde{S}$ of S. The corresponding results for the symplectic group $Sp(2m, R)$ are obtained as corollaries when $*$ is the identity (see [15] for the generation of $Sp(2m, R)$ by symplectic transvections).