Linear Algebra and its Applications 552 (2018) 166–193

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# Graphical criteria for positive solutions to linear systems ☆

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ARTICLE INFO

*Article history:*
Received 21 September 2017
Accepted 14 April 2018
Available online 18 April 2018
Submitted by S. Kirkland

*MSC:*
92C42
80A30

*Keywords:*
Linear system
Positive solution
Spanning forest
Matrix-tree theorem
Chemical reaction networks
Steady state parameterization

ABSTRACT

We study linear systems of equations with coefficients in a generic partially ordered ring $R$ and a unique solution, and seek conditions for the solution to be nonnegative, that is, every component of the solution is a quotient of two nonnegative elements in $R$. The requirement of a nonnegative solution arises typically in applications, for example in biology and ecology, where quantities of interest are concentrations and abundances. We provide novel conditions on a labeled multidigraph associated with the linear system that guarantee the solution to be nonnegative. Furthermore, we study a generalization of the first class of linear systems, where the coefficient matrix has a specific block form and provide analogous conditions for nonnegativity of the solution, similarly based on a labeled multidigraph. The latter scenario arises naturally in chemical reaction network theory, when studying full or partial parameterizations of the positive part of the steady state variety of a polynomial dynamical system in the concentrations of the molecular species.



☆ This work was funded by the Danish Research Council and the Lundbeck Foundation, Denmark.

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https://doi.org/10.1016/j.laa.2018.04.014
0024-3795/

## 1. Introduction

A classical problem in applied mathematics is to determine the solutions to a linear system of equations. In applications, it is often the case that only positive or nonnegative real solutions to the system are meaningful, and criteria to assert positivity and nonnegativity of the solutions have thus been developed [1,13,2,18]. Nonnegativity is required, for example, in the case of equilibria concentrations of molecular species in biochemistry [3,4,9], species abundances at steady state in ecology [14], stationary distributions of Markov chains in probability theory [16], and in Birch's theorem for maximum likelihood estimation in statistics [17]. Also in economics and game theory are equilibria often required to be nonnegative [12,8]. Many of these situations arise from considering dynamical systems where the state variables are restricted to the positive or nonnegative orthant.

We consider a linear system of equations $Ax + b = 0$, with $\det A \neq 0$ and where the coefficients of $A, b$ are in a generic partially ordered ring $R$. By adding an extra row to $A$ such that the column sums are zero, we might associate in a natural way a Laplacian $L$ with the linear system and the corresponding (so-called) labeled canonical multidigraph. The components of the solution to the linear system are rational functions on the entries of $A$ and $b$. Their numerator and denominator can, by means of the Matrix-Tree Theorem [24], be expressed as polynomials on the labels of the rooted spanning trees of the canonical multidigraph, and in fact, of any labeled multidigraph with Laplacian $L$. If the multidigraph is what we call a *P-graph* (Definition 1), then we show that the solution to $Ax + b = 0$ is nonnegative. These conditions are readily fulfilled if the off-diagonal elements of $L$ (not only $A$) are nonnegative.

In the applications motivating this work, the P-graph condition is hardly met. This happens for example in the study of biochemical reaction networks, where the system of interest arises from computing the equilibrium points of a dynamical system constrained to certain invariant linear varieties. However, in many instances the solution is nevertheless nonnegative. In the second part of the paper we explore this other scenario. We consider systems of a specific block form, compatible with the application setting, and derive conditions on another (related) multidigraph that ensure nonnegativity of the solution. In this situation, the solution is expressed as a rational function in the labels of rooted spanning forests of the multidigraph. The second scenario is an extension of the generic case given in the first part of the paper.

Typically in applications, the entries of $A, b$ depend on parameters and inputs that cannot be fixed beforehand, but must be treated as "unknown" or symbolic variables. Our approach accommodates this since solutions are given as rational functions in these entries. Alternatively, one might view the parameters as functions, and apply the results for the ring of real valued functions. In this way, we can study the nonnegativity of solutions without fixing parameters and inputs.

One natural application of our results, which motivated this work, is within biochemical reaction network theory and concerns the parameterization of the positive part of