Linear Algebra and its Applications 552 (2018) 239–255

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Structured distance to normality of tridiagonal matrices [☆]

Natália Bebiano [a,1], Susana Furtado [b,c,*]

[a] CMUC and Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal
[b] CEAFEL, Portugal
[c] Faculdade de Economia do Porto, Universidade do Porto, 4200-464 Porto, Portugal

A R T I C L E I N F O

Article history:
Received 19 March 2018
Accepted 19 April 2018
Available online 30 April 2018
Submitted by P. Semrl

MSC:
65F15
65F35
15B05
15B57

Keywords:
Distance
Structured distance
Frobenius norm
Normal matrix
r-Toeplitz matrix
Tridiagonal matrix

A B S T R A C T

In this article we study the distance d, measured in the Frobenius norm, of a tridiagonal matrix T to the set $\mathcal{I}_T$ of similarly structured irreducible normal matrices. The matrices in the closure of $\mathcal{I}_T$ whose distance to T is d are characterized. Known results in the literature for the cases in which T is real or a Toeplitz matrix are recovered. In addition, the special case in which T is a 2-Toeplitz matrix is considered.

☆ This work was partially supported by project UID/MAT/04721/2013.

* Corresponding author at: Faculdade de Economia do Porto, Universidade do Porto, 4200-464 Porto, Portugal.

E-mail addresses: bebiano@mat.uc.pt (N. Bebiano), sbf@fep.up.pt (S. Furtado).

[1] This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

https://doi.org/10.1016/j.laa.2018.04.023

1. Introduction

Nearness problems are being investigated by several researchers, see, e.g., [1,2,4,8,9,11–15,17,18], and the references therein. In particular, the distance of a given matrix to normality is of interest because the eigenproblem is well conditioned for normal matrices. A small distance to normality may allow the replacement of the considered matrix by a closest normal matrix and the computation of the eigenvalues of the latter.

Many problems in physics, engineering and other sciences are modeled by tridiagonal matrices. In several situations these matrices exhibit further special structure, such as symmetry, break of reflection or time symmetry, or an r-Toeplitz structure (see e.g. [7,13–16] and the references therein). Recall that a tridiagonal r-Toeplitz matrix has r-periodically repeated entries along the sub, main and supra diagonals. For many such structured tridiagonal matrices, as the r-Toeplitz tridiagonal matrices with $r > 2$, or $r = 2$ and even size, there are no explicit formulas for their eigenvalues, see e.g. [5]. So it may be demanded to solve the eigenproblem for nearby normal tridiagonal matrices. Due to Corollary 2, the eigenvalues of a tridiagonal normal matrix can be obtained by calculating the eigenvalues of a Hermitian matrix, a procedure that is well studied and for which there are efficient algorithms.

The main purpose of this paper is to study the distance, measured in the Frobenius norm (see [10] for details), of a complex tridiagonal matrix T to the set of similarly structured irreducible normal matrices. We also determine normal tridiagonal matrices at this distance to T. Though the proofs of our results are very technical and involve tedious computations, the obtained formulae are simple and aesthetically appealing.

The main result in this paper complements and generalizes some known results in the literature. Namely, in [13] the authors considered the case in which T is a real matrix and in [14] the case in which T is a complex tridiagonal Toeplitz matrix. In this note we obtain these results as corollaries of our main result and also obtain a new corollary for the case in which T is a 2-Toeplitz tridiagonal matrix. The general case of r-Toeplitz matrices can be obtained similarly, though it was not explicitly deduced here since it requires some heavy computations.

This paper is organized as follows. In Section 2 a characterization of normal tridiagonal irreducible matrices is given. The main results are Theorems 4 and 5 in Section 3, which are illustrated with a few examples. Section 4 concerns r-Toeplitz tridiagonal matrices. In Section 5 we prove Theorems 4 and 5. Finally, in Section 6 some concluding remarks are presented.

We next summarize the notation used in the paper for some relevant subsets of $\mathbb{C}^{n\times n}$.

$\mathcal{T}$	subset of $\mathbb{C}^{n\times n}$ formed by nonscalar tridiagonal matrices
$\mathcal{I}_T$	subset of $\mathbb{C}^{n\times n}$ formed by irreducible normal tridiagonal matrices
$\mathcal{N}_T$	closure of $\mathcal{I}_T$
$\mathcal{T}_r$	subset of $\mathcal{T}$ formed by r-Toeplitz matrices