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On the values of the permanent of (0,1)-matrices

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ABSTRACT

In this paper we discuss the values of the permanent of (0,1)-matrices of size n. Classical Brualdi–Newman theorem asserts that every integer value from 0 up to 2^{n-1} can be realized as the permanent of such a matrix. We obtain a result which is at least twice better and in particular we show that all nonnegative integer values which are less than or equal to 2^n can be realized. We also investigate the set of integer values that the permanent function cannot attain on the set of $(0, 1)$-matrices.

1. Introduction

Let $\mathbb{F}$ be a field of characteristic zero and M_n be $n \times n$ matrix algebra over $\mathbb{F}$. Following [1] we denote by $\mathfrak{A}_n \subset M_n$ the subset of (0,1)-matrices, that is, matrices with entries from $\{0, 1\} \subseteq \mathbb{F}$.

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Definition 1.1. The *permanent* of $A = (a_{ij}) \in M_n$ is the value

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \dots a_{n\sigma(n)},$$

where S_n denotes the group of permutations on the set $\{1, \dots, n\}$.

Note that, for matrices from $\mathfrak{A}_n$, the values of the permanent are integers with the natural order. Note also that the permanent of a matrix doesn't change under transposition of the matrix and its row or column permutations.

Given $n \in \mathbb{N}$, it is straightforward to check that the maximum possible value of the permanent, considered as a function on $\mathfrak{A}_n$, is $n!$. This value is attained at the matrix J_n such that all of its entries are equal to 1. Note that the values 0 and 1 are also realizable (on zero and identity matrices, respectively). However, for example, the value $n!-1$ is not realizable as the permanent of a certain (0,1)-matrix, since if a matrix contains at least one zero then its permanent is not greater than $n! - (n-1)!$ (at least $(n-1)!$ summands among the $n!$ terms in permanent decomposition turn to be zero). At the same time the following theorem by Brualdi and Newman asserts that there are big sequences of successive integers that are realizable as the permanents of matrices from $\mathfrak{A}_n$.

Theorem 1.2. (Brualdi, Newman, 1965, [1]). *For all $n \in \mathbb{N}$ for any nonnegative integer $j \leq 2^{n-1}$ there exists a matrix $A \in \mathfrak{A}_n$ with* $\operatorname{per}(A) = j$.

A natural open problem is to find out what particular integers from $[0, n!]$ are realizable as the permanents of matrices from $\mathfrak{A}_n$, see [2, Problem 7, p. 230]. In particular, what is the minimal non-realizable value? To discuss the last question further we need the following definition.

Definition 1.3. Let $n \in \mathbb{N}$. We call $B_n \in [0, n!]$ *an upper bound for consecutive values* of the permanent of matrices from $\mathfrak{A}_n$ if B_n is the minimal positive integer with the property that there is no $A \in \mathfrak{A}_n$ such that $\operatorname{per}(A) = B_n + 1$.

Theorem 1.2 provides a lower bound for B_n, i.e. $B_n \geq 2^{n-1}$.

The aim of the present paper is to improve this bound and to prove that $B_n \geq \frac{67}{64} 2^n$ for $n \geq 6$. Since $\frac{67}{64} > 1$, the following corollary holds immediately.

Corollary 1.4. *For all $n \in \mathbb{N}, n \geq 6$, it holds that $B_n \geq 2^n$.*

Then we investigate the values of the permanent which cannot be attained.

Our paper is organized as follows. In Section 2 we introduce the notion of resembling matrices, which are very useful to produce consecutive values of the permanent. In Section 3 we compute the permanent function for some important resembling matrix chains. Section 4 is devoted to the proof of the main result. In Section 5 we investigate non-realizable values of the permanent.