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Moore–Penrose inverse of the incidence matrix of a distance regular graph

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ABSTRACT

Let Γ be a graph with n vertices, where each edge is given an orientation and let Q be the vertex-edge incidence matrix. We obtain a formula for the Moore–Penrose inverse of Q, when Γ is a distance regular graph. The formula is illustrated by examples.

1. Introduction

For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ the vertex set and the edge set of Γ respectively. Let Γ be a graph with $V(\Gamma) = \{1, 2, \ldots, n\}$, $E(\Gamma) = \{e_1, e_2, \ldots, e_m\}$ and suppose each edge of Γ is assigned an orientation. The vertex-edge incidence matrix of Γ, denoted by $Q(\Gamma)$, (or simply by Q if there is no possibility of a confusion) is the $n \times m$ matrix

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defined as follows. The rows and the columns of Q are indexed by $V(\Gamma)$, $E(\Gamma)$ respectively. The (i,j)-entry of Q is 0 if vertex i and edge e_j are not incident and otherwise it is 1 or -1 according as e_j originates or terminates at i respectively.

If A is an $n \times m$ matrix, then the $m \times n$ matrix G is called a generalized inverse of A if $AGA = A$. The Moore–Penrose inverse of A, denoted by A^+, is an $m \times n$ matrix satisfying the following equations [20,21]:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)' = AA^+, \quad (A^+A)' = A^+A.$$

It is well-known that any complex admits a unique Moore–Penrose inverse. We refer to [14,8] for basic properties of the Moore–Penrose inverse.

The Laplacian matrix of Γ, denoted by $L(\Gamma)$, is the $n \times n$ matrix defined as follows. The rows and columns of $L(\Gamma)$ are indexed by $V(\Gamma)$. If $i \neq j$ then the (i,j)-entry of $L(\Gamma)$ is 0 if vertex i and j are not adjacent, and it is -1 if i and j are adjacent. The (i,i)-entry of $L(G)$ is d_i, the degree of the vertex i, $i = 1, 2, \ldots, n$. Then observe that $L(\Gamma) = Q(\Gamma)Q(\Gamma)'$. It is well known that for any matrix A, $(AA')^+ = (A^+)'A^+$. Thus, using formula for Q^+ we may obtain an expression for L^+. The Moore–Penrose inverse of Laplacian matrix is exploited in various applications as the Weiner and Kirchhoff indices, as well as the algebraic connectivity of the graph, moreover it is used for analysis of electrical networks [17]. If we consider a graph Γ to be an electrical network, where each edge is a one-ohm resistor, then we can define the effective resistance between any two vertices i and j as the voltage of a battery which, when connected to the two vertices, causes a current of 1 ampere to flow. The effective distance $r(i,j)$ between the vertices i and j is defined as

$$r(i,j) = L^+_{ii} + L^+_{jj} - L^+_{ij} - L^+_{ji}.$$

This concept was first introduced by Sharpe [22], and then rediscovered by Gvishiani and Gurvich [12], as well as Klein and Randic [16], the reader is referred to [1,2,9,10,13,23,24] for more information. Kirkland et al. in [18] show a relationship between entries of L^+ and various distance functions on trees.

We refer the interested reader to the articles [3,5,6,15] that consider Moore–Penrose inverse of some matrices associated to graphs.

The class of distance regular graphs is an important class of regular graphs while studying several topics such as combinatorial designs, symmetric nets and Hadamard matrices [4,7,11]. The main goal of this paper is to obtain a graph-theoretic description of the Moore–Penrose inverse of the incidence matrix of a distance regular graph. We remark that the entries of the Moore–Penrose inverse of the Laplacian matrices for generalized Johnson graphs is obtained in the recent paper [19]. We illustrate our results using several classical examples of distance regular graphs. Families of distance transitive graphs, which are a subclass of distance regular graphs, are also considered.