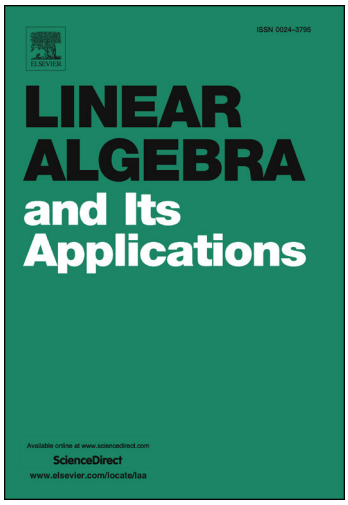

Tropical compound matrix identities

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TROPICAL COMPOUND MATRIX IDENTITIES

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Abstract. We prove identities on compound matrices in extended tropical semirings. Such identities include analogues to properties of conjugate matrices, powers of matrices and $\mathrm{adj}(A)\det(A)^{-1}$, all of which have implications on the eigenvalues of the corresponding matrices. A tropical Sylvester-Franke identity is provided as well.

Keywords: Tropical linear algebra; characteristic polynomial; compound matrix; eigenvalues; permanent; definite matrices; pseudo-inverse.

AMSC: 15A15 (Primary), 15A09, 15A18, 15A24, 15A29, 15A75, 15A80, 15B99.

1. Introduction

The max-plus or tropical semiring $\mathbb{R}_{\max}$ is the set of real numbers $\mathbb{R}$, completed with the element $-\infty$, and equipped with the operations $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$ (also denoted as ab). The zero-element of this structure is therefore $-\infty$, which is its minimal element. See for instance [BCOQ92, But10, MS15, ABG07] and the references therein for more background on linear algebra over the max-plus semiring.

The lack of additive inverses is a source of difficulties in the study of tropical structures. In particular, the notion of "vanishing" has to be adapted: a tropical polynomial vanishes at a point if the maximum of the values of its monomials, evaluated at this point, is achieved twice at least. In applications coming from real geometry [Vir01], one considers tropical polynomials enriched with a sign information. Then, vanishing tropically means that the maximum of the value of the monomials with a positive tropical sign coincides with the maximum of the value of the monomials with a negative tropical sign. In this way, one can define the notion of polynomial identity over the tropical semiring. Such polynomial identities can often be proved by direct combinatorial methods, i.e., by "bijective proofs", along the lines of [Str83, Zei85] or of [Gon83] (see also [GM84]). It was observed in [RS84] that certain determinantal identities over semirings can be derived from their classical analogues, avoiding the recourse to bijective proofs. This idea led to a transfer principle in [Gau92], later refined in [AGG09]. As an application of the transfer principle, a number of determinantal identities (Laplace type expansions [RS84, Plu90, Gau92, AGG09], Binet-Cauchy theorem [Gau92, GBCG98, AGG09]) or more advanced polynomial identities (Amitsur-Levitzki [Gau96, AGG09]), were obtained, with several applications. Polynomial identities have also appeared more recently in works on the "supertropical" extension of the tropical semiring [IR11a, IR11b].

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