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Numerical radius inequalities via convexity

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The main goal of this article is to present numerical radius inequalities for matrices based on convexity of certain numerical radius functions. This simple approach extends some unitarily invariant norms inequalities, such as Heinz and Young inequalities, to the context of numerical radius. However, due to weak unitary invariance of the numerical radius, these extended inequalities will be weaker than the corresponding unitarily invariant norms versions.

1. Introduction

As usual, $\mathbb{M}_n$ will stand for the algebra of all $n \times n$ complex matrices that contains the cone of positive semi-definite matrices $\mathbb{M}_n^+$ and that of strictly positive matrices $\mathbb{M}_n^{++}$. If $A \in \mathbb{M}_n^+$, we write $A \geq 0$ while $A > 0$ will mean $A \in \mathbb{M}_n^{++}$.

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Matrices and their inequalities have attracted researchers working in operator theory. These inequalities have been studied in different approaches among which unitarily invariant norms are most common. In this context, a unitarily invariant norm is a norm $|||\ \ |||$ defined on $\mathbb{M}_n$ with the additional property $|||UAV||| = |||A|||$ for all unitaries $U, V \in \mathbb{M}_n$. We refer the reader to [3,4,10,14–18] as a sample of research treating inequalities governing unitarily invariant norms.

Among the most basic inequalities for unitarily invariant norms are

$$2|||A^{1/2}XB^{1/2}||| \le |||A^tXB^{1-t} + A^{1-t}XB^t||| \le |||AX + XB||| \tag{1.1}$$

where $A, B \in \mathbb{M}_n^+$, $X \in \mathbb{M}_n$ and $0 \le t \le 1$. This pair of inequalities is usually referred to as Heinz inequality; in which the middle term are the Heinz means which interpolate between the geometric and arithmetic means.

On the other hand, the Hölder-type inequality

$$|||A^tXB^t||| \le |||X|||^{1-t}|||AXB|||^t \tag{1.2}$$

was proved in [11] for the same parameters. This inequality then entails the Young-type inequality

$$|||A^tXB^{1-t}||| \le t|||AX||| + (1-t)|||XB|||. \tag{1.3}$$

A stronger version than (1.3) would be

$$|||A^tXB^{1-t}||| \le |||t\ AX + (1-t)XB|||. \tag{1.4}$$

Unfortunately, (1.4) is not true for arbitrary unitarily invariant norm. However, it is true for the Hilbert–Schmidt norm $\|\ \ \|_2$, see [8].

A weaker version is having $X = I$, the identity, in (1.4). In this case, the inequality is true as proved first in [3].

The main purpose of this paper is to study similar inequalities for the numerical radius. Recall that the numerical radius $\omega(A)$ of the matrix $A \in \mathbb{M}_n$ is defined as

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

It is well known that ω defines a norm on $\mathbb{M}_n$. However, this norm is not unitarily invariant, but is weakly invariant, meaning that $\omega(UAU^*) = \omega(A)$ for any unitary matrix U.

In [12] the Heinz-type inequality

$$\omega\left(A^{1/2}XB^{1/2}\right) \le \frac{1}{2}\omega\left(A^tXB^{1-t} + A^{1-t}XB^t\right), A, B \ge 0, X \in \mathbb{M}_n \tag{1.5}$$