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# A generalization of the Kreiss Matrix Theorem

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ABSTRACT

Let $\Omega$ be a compact subset of the complex plane such that its complement is simply connected in the extended complex plane. Suppose $A$ is a linear bounded operator in a Hilbert space, with spectrum $\sigma(A) \subset \Omega$. If $\Omega$ is symmetric with respect to the real line and $f$ is a Markov function, we show that

$$\|f(A)\| \leq \mathrm{e}\, C\, \mathcal{K}(\Omega)\ \|f\|_\Omega ,$$

where $\mathcal{K}(\Omega)$ is the Kreiss constant with respect to $\Omega$ and $C$ is a constant. We also present other extensions of the Kreiss Matrix Theorem to arbitrary holomorphic functions.


## 1. Introduction and preliminary results

Let $A$ be an $N \times N$ complex matrix with spectrum $\sigma(A)$, and let $\|\cdot\|$ denotes the operator norm which is defined with respect to the usual Euclidean norm on vectors. The original statemet of the Kreiss Matrix Theorem (KMT), a well known fact in applied

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matrix analysis, characterizes power bounded and exponentially bounded matrices. If $\sigma(A)$ is in the closed unit disc $\overline{\mathbb{D}}$, this celebrated theorem is known by the following inequality

$$\mathcal{K}(A) \leq \sup_{n\geq 0} \|A^n\| \leq e\, N\, \mathcal{K}(A), \tag{1}$$

where

$$\mathcal{K}(A) := \sup_{|z|>1} (|z|-1) \left\| \left(zI - A\right)^{-1} \right\|$$

is *the Kreiss constant* with respect to the closed unit disc. If $\sigma(A)$ is in the left half-plane, then the analogous result for matrix exponentials is the following inequality

$$\mathcal{K}(A) \leq \sup_{t\geq 0} \left\|e^{tA}\right\| \leq e\, N\, \mathcal{K}(A), \tag{2}$$

where here $\mathcal{K}(A)$ is the Kreiss constant with respect to the closed left half-plane and defined by

$$\mathcal{K}(A) := \sup_{Re(z)>0} \big(Re(z)\big) \left\| \left(zI - A\right)^{-1} \right\|.$$

The constant e$N$, in the upper bounds in the inequalities (1) and (2), was obtained after several developments of the original statement in 1962, see [5]. The original proof shows the upper bound $\mathcal{K}(A)^{N^N}$. Then improvements were made by Morton in 1964, Strang in 1966 and Miller in 1967, which decreased the upper bound to $6^N(N+4)^{5N}\mathcal{K}(A)$, $N^N\mathcal{K}(A)$ and $\mathrm{e}^{9N^2}\,\mathcal{K}(A)$, respectively. From a paper by Laptev, Tadmor showed in 1981 that this upper bound is linear in $N$ and reduced it to $32\,\mathrm{e}\,N\mathcal{K}(A)$. Leveque and Trefethen minimized this upper bound in 1984 to $2\,\mathrm{e}\,N\mathcal{K}(A)$ and they conjectured that the optimal bound is $\mathrm{e}\,N\mathcal{K}(A)$. In 1985, Smith reduced this upper bound to $(1+\frac{2}{\pi})\,\mathrm{e}\,N\mathcal{K}(A)$ and finally Spijker proved the conjecture in 1991. The proof of the optimal upper bound is based on a lemma, which provides an upper bound for the arc length of the image of the unit circle in the complex plane by a rational function. For details, see [6,8,10] and their references therein.

Many applications in science require the evaluation of the norm of $f(A)$ for an arbitrary holomorphic function $f$ on a neighborhood of the spectrum of $A$. An interesting connection between $f(A)$ and the resolvent $(zI-A)^{-1}$ is given by the Cauchy integral formula. So, it is important to study the norm of the resolvent to understand $f(A)$. Recently, it has been shown that the norm of the resolvent determines the norm behavior of $f(A)$ if $f$ is a Möbius transformation. However, it is not sufficient for general holomorphic functions, see [7] for details. The idea of the generalization of KMT is to get an appropriate upper bound of $\|f(A)\|$ if we have an appropriate upper bound of the resolvent norm. In [9], KMT was extended to Faber polynomials on a general complex