Linear Algebra and its Applications 549 (2018) 100–122

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The computation of the mean first passage times for Markov chains

Jeffrey J. Hunter

Department Mathematical Sciences, School of Engineering, Computer and Mathematical Sciences, Auckland University of Technology, Private Bag 92006, Auckland 1142, New Zealand

ARTICLE INFO

Article history:
Received 30 December 2016
Accepted 6 March 2018
Available online 12 March 2018
Submitted by S. Kirkland

MSC:
15A09
15B51
60J10

Keywords:
Markov chain
Stochastic matrix
Moments of first passage times
Generalized matrix inverses

ABSTRACT

A survey of a variety of computational procedures for finding the mean first passage times in Markov chains is presented. The author recently developed a new accurate computational technique, an Extended *GTH* Procedure, Hunter (2016) [17] similar to that developed by Kohlas (1986) [20]. In addition, the author recently developed a variety of new perturbation techniques for finding key properties of Markov chains including finding the mean first passage times, Hunter (2016) [18]. These recently developed procedures are compared with other procedures including the standard matrix inversion technique using the fundamental matrix Kemeny and Snell (1960) [19], some simple generalized matrix inverse techniques developed by Hunter (2007) [15], and some modifications to the *FUND* technique of Heyman (1995) [7]. *MATLAB* is used to compute errors and estimate computation times when the techniques are used on some test problems that have been used in the literature together with some large sparse state-space cases. For accuracy a preference for the procedure of the author is exhibited for the test problems. However it appears that the procedure, as presented, requires longer computational times.

E-mail address: jeffrey.hunter@aut.ac.nz.

https://doi.org/10.1016/j.laa.2018.03.010

1. Introduction

In Markov chain (MC) theory mean first passage times ($MFPT$s) provide significant information regarding the short term behaviour of the MC. A review of $MFPT$s, together with details regarding stationary distributions and the group inverse of the Markovian kernel, is given in [18]. We refer the reader to this aforementioned article as it provides the relevant background to this paper and enables us to avoid repetition of the material. In Hunter [18], which focuses on computational techniques for the key properties of irreducible MCs using perturbation techniques, we commented that in a sequel paper we would consider a variety of other techniques to get a better impression as to whether perturbation procedures may in fact prove to be suitable alternatives. We address these issues in this paper.

We firstly set the scene by reintroducing the notation that was used in [18].

Let $\{X_n, n \geq 0\}$ be a finite MC with state-space $S = \{1, 2, \ldots, m\}$ and transition matrix $P = [p_{ij}]$, where $p_{ij} = P\{X_n = j | X_{n-1} = i\}$ for all $i, j \in S$.

The stationary distribution $\{\pi_j\}$, $(1 \leq j \leq m)$, exists and is unique for all irreducible *MCs*, that $\pi_j > 0$ for all j, and satisfies the equations (the *stationary equations*)

$$\pi_j = \sum_{i=1}^{m} \pi_i p_{ij} \quad \text{with} \quad \sum_{i=1}^{m} \pi_j = 1. \tag{1.1}$$

If $\boldsymbol{\pi}^T \equiv (\pi_1, \pi_2, \ldots, \pi_m)$, the *stationary probability vector*, and $\boldsymbol{e}$ is a column vector of 1's, the stationary equations (1.1) can be expressed as

$$\boldsymbol{\pi}^T (I - P) = \boldsymbol{0}^T, \quad \text{with } \boldsymbol{\pi}^T \boldsymbol{e} = 1. \tag{1.2}$$

Let $T_{ij} = \min[n \geq 1, X_n = j | X_0 = i]$ be the first passage time from state i to state j (first return when $i = j$) and define $m_{ij} = E[T_{ij} | X_0 = i]$ as the $MFPT$ from state i to state j (or mean recurrence time of state i when $i = j$). For finite irreducible *MCs* all the m_{ij} are well defined and finite. Let $M = [m_{ij}]$ be the $MFPT$ matrix. Let $\boldsymbol{\delta}_{ij} = 1$, when $i = j$ and 0, when $i \neq j$. Let $M_d = [\boldsymbol{\delta}_{ij} m_{ij}]$ be the diagonal matrix formed from the diagonal elements of M, and $E = [1]$ (i.e. all the elements are unity).

It is well known ([19]) that, for $1 \leq i, j \leq m$,

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}. \tag{1.3}$$

In particular, for all $j \in S$, the mean recurrence time of state j is given by

$$m_{ij} = 1/\boldsymbol{\pi}_j. \tag{1.4}$$

From (1.3) and (1.4) it follows that M satisfies the matrix equation