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Extreme points and identification of optimal alternate dual frames

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ABSTRACT

The main purpose of this paper is to identify the set of optimal dual frames under erasures. To this end, we characterize extreme points in the set of all optimal duals for 1-erasure. Also, we give some conditions under which an alternate dual frame is either not optimal or is a non-unique optimal dual. Moreover, we obtain a new characterization of frames when the canonical dual is the unique optimal dual frame for erasures.

1. Introduction and preliminaries

Frames are redundant systems of vectors in separable Hilbert spaces, which present various decompositions for the elements in Hilbert spaces. This considerable property leads to significant applications in various areas of pure and applied science that have

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been analyzed by many researchers in over the past two decades [1–5,7]. Indeed, in frame theory every $f \in \mathcal{H}$ can be represented by the measurements $\{\langle f, f_i\rangle\}_{i\in I}$. From these measurements f can be recovered using a reconstruction formula by a dual frame, i.e. a sequence $\{g_i\}_{i\in I}$ for which $f = \sum_{i\in I}\langle f, f_i\rangle g_i$. In real applications, in these transmissions usually a part of the data vectors are reshaped or erased, and we need to perform the reconstruction by using the partial information at hand. Hence, one of the deepest and most precious problems in frame theory is optimal dual problem, which asks for finding the best dual frames that minimize the reconstruction errors when erasures occur. This concept first was considered by Han et al. in [9,10]. In this paper we continue the investigation on finding optimal dual frames, in particular alternate dual frames. Moreover, we survey this problem from a new point of view by extreme points. First, we review some preliminaries of frame theory.

Let $\mathcal{H}$ be a separable Hilbert space and I a countable index set. A sequence $F := \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist the positive constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.1}$$

The constants A and B are called the *frame bounds*. If A and B can be chosen such that $A = B$, the frame F is called a *tight frame*, and in the case of $A = B = 1$ it is a *Parseval frame*. For a frame $\{f_i\}_{i\in I}$, the *synthesis operator* $T_F : l^2 \to \mathcal{H}$ is defined by $T_F\{c_i\} = \sum_{i\in I} c_i f_i$. If $\{f_i\}_{i\in I}$ is a frame, then $S_F = T_F T_F^*$ where $T_F^* : \mathcal{H} \to l^2$ the adjoint of T, given by $T_F^* f = \{\langle f, f_i\rangle\}_{i\in I}$, is called the *analysis operator*. A frame $G := \{g_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *dual* for $\{f_i\}_{i\in I}$ if $T_G T_F^* = I_{\mathcal{H}}$. A special dual frame is $\{S_F^{-1} f_i\}_{i\in I}$, which is called the canonical dual of F [6]. It is well known that $\{g_i\}_{i\in I}$ is a dual of frame $\{f_i\}_{i\in I}$ if and only if $g_i = S_F^{-1} f_i + u_i$, for all $i \in I$ where $\{u_i\}_{i\in I}$ satisfies

$$\sum_{i\in I}\langle f, u_i\rangle f_i = 0, \quad (f \in \mathcal{H}). \tag{1.2}$$

In this paper we only consider non-zero finite frames in finite dimensional Hilbert spaces. Consider $I_k = \{1, ..., k\}$ and let $F = \{f_i\}_{i\in I_k}$ be a frame for n-dimensional Hilbert space $\mathcal{H}_n$, in this case we call F a (k, n)-frame. If $G = \{g_i\}_{i\in I_k}$ is a dual of F and $J \subset I_k$, then the error operator E_J is defined by

$$E_J f = \sum_{i\in J}\langle f, f_i\rangle g_i = (T_G D T_F^*) f, \quad (f \in \mathcal{H})$$

where D is $k \times k$ diagonal matrix with $d_{ii} = 1$ for $i \in J$ and 0 otherwise. Let

$$d_r(F, G) = \max\{\|T_G D T_F^*\| : D \in \mathcal{D}_r\} = \max\{\|E_J\| : \mathrm{card}(E_J) = r\}, \tag{1.3}$$