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Sequence characterizations of double Riordan arrays and their compressions

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ABSTRACT

Inspired by the Fibonacci tree shown in the recent book, Catalan Numbers, by Richard Stanley, we present a combinatorial way to construct double Riordan arrays by using ECO technique. The sequence characterizations of double Riordan arrays and subgroups of the double Riordan group are found. As an extension of Pascal–Fibonacci triangle, the compression forms of double Riordan arrays called double quasi-Riordan arrays are defined. The connections of lower triangular matrices and double Riordan arrays as well as their compressions are given. Those results are also extended to the case of high order Riordan arrays, which are defined in the paper. The pairs of Sheffer polynomials and pairs of summation formulas associated with double Riordan arrays are defined and discussed.

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1. Introduction

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [23]). More formally, let us consider the ring of formal power series (f.p.s.) $\mathcal{F} = \mathbb{C}[\![t]\!]$ defined on the complex field $\mathbb{C}$. The *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{C}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k. \tag{1}$$

In other words, $d(t)h(t)^k$ is the generating function of the kth column of the Riordan array D.

From the *fundamental theorem of Riordan arrays* (see [21]), $(d, h)f = df \circ h$, it is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1, h_1)(d_2, h_2) = (d_1(d_2 \circ h_1), h_2 \circ h_1). \tag{2}$$

The Riordan array $I = (1, t)$ acts as an identity for this product. Several subgroups of $\mathcal{R}$ are important. The Appell subgroup $\mathcal{A}$ of *Appell arrays* is the collection of all Riordan arrays $R = (d(t), t)$ in $\mathcal{R}$. The associate subgroup $\mathcal{L}$ of *Lagrange arrays* is the collection of all Riordan arrays $R = (1, h(t))$ in $\mathcal{R}$. The Bell subgroup $\mathcal{B}$ of *Bell* or *renewal arrays* is the collection of all Riordan arrays $R = (d(t), td(t))$ in $\mathcal{R}$. A Riordan array (d, h) can be described by using its row relationship. It is well known (see, for instance, [16,15,12]) that for all entries lying on kth column ($k \geq 1$), there holds

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n} \tag{3}$$

for $k \geq 0$, where the coefficient sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ is refereed *A-sequence*. Hence, the generating function of A-sequence satisfying

$$h(t) = tA(h(t)). \tag{4}$$

There also exists a unique *Z-sequence* $Z = (z_0, z_1, z_2, \ldots)$ such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{5}$$

or equivalently (see [12]),

$$d(t) = \frac{d_{0,0}}{1 - tZ(h(t))}, \tag{6}$$