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A bound on the spectral radius of hypergraphs with e edges

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ABSTRACT

For $r \geq 3$, let $f_r\colon [0, \infty) \to [1, \infty)$ be the unique analytic function such that $f_r(\binom{k}{r}) = \binom{k-1}{r-1}$ for any $k \geq r - 1$. We prove that the spectral radius of an r-uniform hypergraph H with e edges is at most $f_r(e)$. The equality holds if and only if $e = \binom{k}{r}$ for some positive integer k and H is the union of a complete r-uniform hypergraph K_k^r and some possible isolated vertices. This result generalizes the classical Stanley's theorem on graphs.

1. History

The spectral radius $\rho(G)$ of a graph G is the maximum eigenvalue of its adjacency matrix. Which graph has the maximum spectral radius among all graphs with e edges? If $e = \binom{k}{2}$, Brualdi and Hoffman [1] proved that the maximum of $\rho(G)$ is reached by

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the union of a complete graph on k vertices and some possible isolated vertices. They conjectured that the maximum spectral radius of a graph G with $e = \binom{k}{2} + s$ edges is attained by the graph G_e, which is obtained from complete graph K_k by adding a new vertex and s new edges. In 1987, Stanley [18] proved that the spectral radius of a graph G with e edges is at most $\frac{\sqrt{1+8e}-1}{2}$. The equality holds if and only if $e = \binom{k}{2}$ and G is the union of the complete graph K_k and some isolated vertices. Friedland [7] proved a bound which is tight on the complete graph with one, two, or three edges removed or the complete graph with one edge added. Rowlinson [17] finally confirmed Brualdi and Hoffman's conjecture, and proved that G_e attains the maximum spectral radius among all graphs with e edges.

On the problem of maximizing spectral radius of a certain class of hypergraphs, Fan, Tan, Peng and Liu [6] determined the extremal spectral radii of several classes of r-uniform hypergraphs with few edges. Xiao, Wang and Lu [19] determined the unique r-uniform supertrees with maximum spectral radii among all r-uniform supertrees with given degree sequences. Li, Shao, and Qi [13] determined the extremal spectral radii of r-uniform supertrees. In [9], Kang, Liu, Qi, and Yuan solved a conjecture of Fan et al. [6] related to compare the spectral radii of some 3-uniform hypergraphs. Chen, Chen, and Zhang [4] proved several good upper bounds for the adjacency and signless Laplacian spectral radii of uniform hypergraphs in terms of degree sequences.

In this paper, we will generalize Stanley's theorem to hypergraphs, that is, maximizing the spectral radius of r-uniform hypergraphs among all r-uniform hypergraphs with a given number of edges. For $r \geq 3$, an r-uniform hypergraph H on n vertices consists of a vertex set V and an edge set $E \subseteq \binom{V}{r}$. Cooper and Dutle [5] defined the adjacency tensor A of H to be the r-order n-dimensional tensor $A = (a_{i_1 \cdots i_r})$ by

$$a_{i_1 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!} & \text{if } \{i_1, \ldots, i_r\} \text{ is an edge of } H, \\ 0 & \text{otherwise,} \end{cases}$$

where each i_j runs from 1 to n for $j \in [r]$. The adjacency tensor A of r-uniform hypergraph is always nonnegative and symmetric.

Given an r-uniform hypergraph H, the polynomial form $P_H(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}$ is defined for any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$P_H(\mathbf{x}) = \sum_{i_1,\ldots,i_r=1}^{n} a_{i_1 \cdots i_r} x_{i_1} \cdots x_{i_r} = r \sum_{\{i_1,\ldots,i_r\} \in E(H)} x_{i_1} \cdots x_{i_r}.$$

Then the spectral radius of an r-uniform hypergraph H is

$$\rho(H) = \max_{\|\mathbf{x}\|_r = 1} P_H(\mathbf{x}) = \max_{\|\mathbf{x}\|_r = 1} r \sum_{\{i_1,\ldots,i_r\} \in E(H)} x_{i_1} \cdots x_{i_r},$$

where $\|\mathbf{x}\|_r = \left(\sum_{i=1}^{n} |x_i|^r \right)^{1/r}$.