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On switching classes of graphs

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ABSTRACT

A class of non-oriented simple graphs is called Seidel switching self-complementary (s.s.c. for short) if the complement of any representing graph is in the same equivalence class. Relating the Seidel adjacency matrix of a graph with a Gram matrix, we introduce the 3-signature (s, t) of a switching class of n-vertex graphs. The numbers s and t are the numbers of positive and negative triples within any representing graph of the class. It appears that, for any s.s.c. class of n-vertex graphs, these numbers are equal, yielding $\binom{n}{3}$ even. Consequently if $n \equiv 3 \pmod 4$ then there is no s.s.c. class of n-vertex graphs. We also prove that this 3-signature depends only on the spectrum of the adjacency matrix of the graph. We then consider the switching classes of Paley conference graphs with $4k+2$ vertices, $4k+1 = p^\alpha$, p an odd prime and α a positive integer. We reprove that these classes are s.s.c. Moreover, it is proven that all $4k$-vertex graphs contained in a $(4k+2)$-vertex Paley conference graph are switching equivalent and their class is still a s.s.c. class. In addition, the 3-signature is generalized in view of obtaining a complete invariant of switching classes up to order 8.

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1. Introduction

We deal with simple non-oriented graphs. Two graphs are called isomorphic if there is a one-to-one correspondence between their vertex-sets which preserves adjacency. The complement $\overline{G}$ of a graph G has the same vertex-set as G but two vertices of $\overline{G}$ are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph is self-complementary if it is isomorphic to its complement. There is a huge literature about self-complementary graphs, see e.g. [4,5,13,14] and the references therein.

In this article, we are concerned with another notion of self-complementarity. Given a graph, a *Seidel switching* at any vertex consists of erasing the existing edges and adding the nonexisting ones. Two graphs are called *Seidel switching equivalent* if one can pass from one graph to the other by Seidel switching operations. A *Seidel switching self-complementary class of graphs* (s.s.c. class for short) is a class of graphs in which any representing graph is Seidel switching equivalent to its complement. This notion has been introduced by van Lint and Seidel [9] in relation with equiangular lines of the Euclidean space, see Section 2. This geometric setting yields quickly our first result: With the use of the shape invariant of triples [1,2] we obtain that a necessary condition for a class of n-vertex graphs to be s.s.c. is that $\binom{n}{3}$ be even. This result already appeared in [7], but with different methods. In [15], Sozański determines explicitly the number of s.s.c. classes of n-vertex graphs for any integer n.

In 1933, Paley [11] used the Legendre symbol on finite fields to construct symmetric conference matrices of order $p^\alpha + 1$, p prime, $p^\alpha \equiv 1 \pmod 4$. We call the corresponding graph a *Paley conference graph*. We recall this construction in Section 3. In [14], Sachs proved that the well-known p^α-vertex Paley graph is self-complementary (in the ordinary sense). In [3], the authors quoted without proof that the $(p^\alpha+1)$-vertex Paley conference graph is in a s.s.c. class. In [6] the authors proved that all p^α-vertex subgraphs of the Paley conference graph are switching equivalent, hence they are representing graphs of a s.s.c. class. In this article we pursue this road one step further, proving that all $(p^\alpha - 1)$-vertex subgraphs of a Paley conference graph are also switching equivalent and that their class is s.s.c.

In Section 4 we introduce a more general notion of signature of a switching class of graphs and we prove that this signature together with the determinant form a complete invariant up to order 8 at least for the number of vertices. The question whether this remains true for all orders is analogous in our context to the well-known Kelly–Ulam reconstruction conjecture [8,12]. Another related question of reconstruction in the context of vertex-switching was asked by Stanley [17]. Note that these three questions are independent.

2. Equiangular lines and Seidel switching classes of graphs

A n-tuple of lines of $\mathbb{R}^d$ is called *equiangular of parameter* $a \in]0,1[$ if, for any two lines and any unit vectors x, y on each of them, the inner product of x and y satisfies