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Elliptical higher rank numerical range of some Toeplitz matrices

Maria Adam [a], Aikaterini Aretaki [a,*], Ilya M. Spitkovsky [b]

[a] *Department of Computer Science and Biomedical Informatics, University of Thessaly, 35131 Lamia, Greece*
[b] *Division of Science and Mathematics, New York University Abu Dhabi (NYUAD), Saadiyat Island, Abu Dhabi, United Arab Emirates*

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The higher rank numerical range is described for a class of matrices which happen to be unitarily reducible to direct sums of (at most) 2-by-2 blocks. In particular, conditions are established under which tridiagonal matrices have elliptical rank-k numerical ranges.

1. Introduction

Researchers from the field of theoretical physics implemented several methodologies to resolve problems arising in quantum error correction. The main effort was to eliminate

* Corresponding author.
E-mail addresses: madam@dib.uth.gr (M. Adam), kathy@mail.ntua.gr (A. Aretaki), ims2@nyu.edu, imspitkovsky@gmail.com (I.M. Spitkovsky).

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the error factors created during transmission of quantum information and to describe possible corruption induced in the quantum system. Motivated by a physical problem, Choi et al. in their pioneering articles [7–9], reduced this problem to a purely mathematical introducing the notion of higher rank numerical ranges, and triggering the interest of many authors leading to an extensive literature [1,2,16,17,21].

Let $\mathcal{M}_{m,n}(\mathbb{C})$ (resp., $\mathcal{M}_{m,n}(\mathbb{R})$) denote the set of all $m \times n$ complex (resp., real) matrices, with the notation $\mathcal{M}_{n,n}(\mathbb{C})$ abbreviated further to $\mathcal{M}_n(\mathbb{C})$.

For a positive integer $1 \leq k \leq n$, the *rank-k numerical range* of $A \in \mathcal{M}_n(\mathbb{C})$ is defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} \colon PAP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P\}.$$

Note that the rank-1 numerical range coincides with the classical *numerical range* [15]

$$\Lambda_1(A) \equiv F(A) = \{x^* Ax : x \in \mathbb{C}^n,\, x^* x = 1\}.$$

The latter set encompasses all the eigenvalues of matrix A, that is the *spectrum* $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$.

The higher rank numerical ranges $\{\Lambda_k(A)\}_{k \geq 1}$ form a decreasing sequence of compact sets, due to the inclusion relations

$$\Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_k(A)$$

and they further enjoy a number of basic algebraic and geometric properties [7,8,16]:

(P$_1$) $\Lambda_k(aA + bI) = a\Lambda_k(A) + b$, for any $a, b \in \mathbb{C}$.

(P$_2$) $\Lambda_k(U^* AU) = \Lambda_k(A)$, for any unitary $U \in \mathcal{M}_n(\mathbb{C})$.

(P$_3$) $\Lambda_k(A) \subseteq \Lambda_k(H(A)) + \mathrm{i}\Lambda_k(S(A))$, where $H(A) = (A + A^*)/2$ and $S(A) = (A - A^*)/2\mathrm{i}$ are the Hermitian and skew-Hermitian parts of matrix A, respectively.

(P$_4$) $\Lambda_k(A_1 \oplus A_2) \supseteq \Lambda_k(A_1) \cup \Lambda_k(A_2)$, where the symbol $\oplus$ stands for the direct sum of matrices $A_1, A_2 \in \mathcal{M}_n(\mathbb{C})$.

(P$_5$) $\Lambda_{k_1+k_2}(A_1 \oplus A_2) \supseteq \Lambda_{k_1}(A_1) \cap \Lambda_{k_2}(A_2)$, for any $k_1, k_2 \in \{1, \ldots, n\}$.

(P$_6$) If $n \geq 3k - 2$, then $\Lambda_k(A) \neq \emptyset$. On the other hand, $\Lambda_n(A) \neq \emptyset$ precisely when $A = \lambda I_n$.

For any $1 \leq k \leq n$, $\Lambda_k(A)$ are convex sets (see [21]). Specifically, they coincide with the intersection of half-planes

$$\Lambda_k(A) = \bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re} z \leq \lambda_k(H_\theta(A))\}, \tag{1.1}$$

where $\lambda_k(\cdot)$ denotes the k-th largest eigenvalue of a matrix and $H_\theta(A) = H(e^{\mathrm{i}\theta} A)$ (see [17]). In case of a normal matrix A with spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, the expression (1.1) yields the intersection of the convex hulls (polygons)