Linear Algebra and its Applications 548 (2018) 1–18

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Duality and the signed Laplacian matrix of a graph

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ARTICLE INFO

Article history:
Received 15 October 2017
Accepted 20 February 2018
Available online 24 February 2018
Submitted by P. Semrl

MSC:
primary 05C10, 15B36
secondary 05C05, 05C22, 05C50

Keywords:
Dual graphs
Signed Laplacian matrix
Unimodular congruence
Kirchhoff polynomials

ABSTRACT

We give a necessary and sufficient condition for a bijection between the edge sets of two graphs to be a dual bijection. The condition involves unimodular congruence of augmented signed Laplacian matrices for the two graphs.

1. Dual graphs and Laplacian matrices

Let G be a connected graph with n vertices $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. Multiple edges and loops are allowed. The *Laplacian matrix*, $L(G)$, of G is the $n \times n$ matrix defined as follows:

$$L(G) = \sum_{e \in E(G)} F(e),$$

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https://doi.org/10.1016/j.laa.2018.02.022

where for edge $e = ij$, $F(e)$ is the $n \times n$ matrix whose only nonzero entries are $+1$ in diagonal positions (i,i) and (j,j) and -1 in positions (i,j) and (j,i). If $e = ii$ is a loop, then $F(e) = 0$, otherwise $F(e)$ is a symmetric rank-one matrix whose rows (and columns) sum to zero.

For example, if G has three vertices and four edges, $E(G) = \{12, 12, 23, 33\}$ then

$$\begin{aligned} L(G) &= F(12) + F(12) + F(23) + F(33) \\ &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \end{aligned}$$

The loop 33 contributes nothing to the Laplacian matrix; adding or removing loops from the edge set of a graph does not change the Laplacian matrix for the graph.

For an excellent survey of the Laplacian matrix, see [2].

We write $A \overset{\text{ucon}}{\cong} B$ if there exists a unimodular matrix U such that $B = UAU^T$.

A *reduced Laplacian matrix* $L'_i(G)$ for G is the $(n-1) \times (n-1)$ matrix obtained by deleting row and column i from $L(G)$. There are n different reduced Laplacian matrices and they are all congruent by a unimodular matrix. (See Lemma 1 in Section 5.) Throughout the rest of this paper, $L'(G)$ denotes any one of the reduced Laplacian matrices of G.

Definition 1 *(Dual bijection).* Two graphs G, H are *duals* if there is a bijection $\delta : E(G) \to E(H)$ such that edge sets of spanning trees in G correspond to the complements of edge sets of spanning trees in H. If there is a bijection satisfying the above condition, it is called a *dual bijection*.

In this paper, we establish necessary and sufficient conditions involving unimodular congruence and Laplacian matrices for a bijection from $E(G)$ to $E(H)$ to be a dual bijection. The first result of this type appeared in [1]. Using methods from knot theory—in particular Goeritz congruence and Reidemeister moves (see [4])—it gives a necessary condition for G and H to be dual graphs.

Proposition 1 *([1], Corollary 1). Let G and H be connected dual planar graphs with n and k vertices, respectively, and let $L'(G), L'(H)$ be (any) reduced Laplacian matrices for G and H. Then there exist $(0, \pm 1)$-diagonal matrices Δ_1, Δ_2 such that*

$$L'(G) \oplus (-\Delta_2) \overset{\text{ucon}}{\cong} \Delta_1 \oplus (-L'(H)). \tag{1}$$