Linear Algebra and its Applications 548 (2018) 19–41

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the singular values of matrices with high displacement rank

Alex Townsend [a,1], Heather Wilber [b,*,2]

[a] Department of Mathematics, Cornell University, Ithaca, NY 14853, United States
[b] Center for Applied Mathematics, Cornell University, Ithaca, NY 14853, United States

ARTICLE INFO

Article history:
Received 15 December 2017
Accepted 26 February 2018
Available online 1 March 2018
Submitted by E. Tyrtyshnikov

MSC:
65F30
65N35
15A18
15A24

Keywords:
Low rank approximation
Sylvester matrix equation
Fast Poisson solver
Alternating direction implicit method
Singular values
Displacement structure

ABSTRACT

We introduce a new ADI-based low rank solver for $AX - XB = F$, where F has rapidly decaying singular values. Our approach results in both theoretical and practical gains, including (1) the derivation of new bounds on singular values for classes of matrices with high displacement rank, (2) a practical algorithm for solving certain Lyapunov and Sylvester matrix equations with high rank right-hand sides, and (3) a collection of low rank Poisson solvers that achieve spectral accuracy and optimal computational complexity.

* Corresponding author.
E-mail addresses: townsend@cornell.edu (A. Townsend), hdw27@cornell.edu (H. Wilber).
[1] This work is supported by National Science Foundation grant No. 1645445.
[2] This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under grant No. DGE-1650441.

https://doi.org/10.1016/j.laa.2018.02.025
0024-3795/© 2018 Elsevier Inc. All rights reserved.

1. Introduction

Matrices with rapidly decaying singular values appear with extraordinary frequency in computational mathematics. Such matrices are said to have low numerical rank, and a collection of low rank approximation methods used in particle simulation [1], reduced-order modeling [2,3] and matrix completion [4] has developed around exploiting them. An explicit bound on the numerical rank of a matrix requires bounding its singular values, and this is generally difficult. However, bounds can be derived for families of matrices that have displacement structure [5–7]. In this paper, we derive explicit bounds on the singular values of matrices with displacement structure in cases where known bounds fail to be informative. Our method is constructive and leads to an efficient low rank approximation scheme that we call the factored-independent alternating direction implicit (FI-ADI) method. It can be used, among other things, to develop fast and spectrally-accurate low rank solvers for certain elliptic partial differential equations (PDEs).

The matrix $X \in \mathbb{C}^{m\times n}$ satisfying the Sylvester matrix equation

$$AX - XB = F, \quad A \in \mathbb{C}^{m\times m}, \quad B \in \mathbb{C}^{n\times n}, \tag{1}$$

is said to possess displacement structure, with an (A, B)-displacement rank of $\rho = \operatorname{rank}(F)$. Explicit bounds on the singular values of structured matrices, such as Cauchy, Löwner, real positive definite Hankel and real Vandermonde matrices, can be derived by observing that these matrices satisfy (1) for some specific triple (A, B, F), with F of rank 1 or 2 [5]. Other work has focused primarily on the case where (1) is a Lyapunov matrix equation (i.e., $B = -A^*$, $F = F^*$, with M^* denoting the Hermitian transpose of M) and $\operatorname{rank}(F) = 1$. Closed-form solutions, approximation by exponential sums, Cholesky factorizations, and the convergence properties of iterative methods have been used to derive bounds on the singular values of X [8,9,7,10,11].

Rapidly decaying singular values imply that X is well approximated by a low rank matrix.

Definition 1. Let $X \in \mathbb{C}^{m\times n}$, $m \geq n$, and $0 < \epsilon < 1$ be given. The ϵ-rank of X, denoted by $\operatorname{rank}_\epsilon(X)$, is the smallest integer k such that

$$\sigma_{k+1}(X) \leq \epsilon \, \|X\|_2,$$

where $\sigma_j(X)$ denotes the jth singular value of X, $\sigma_1(X) = \|X\|_2$, and $\sigma_j(X) = 0$ for $j > n$.

In this paper, we relax the assumption that $\operatorname{rank}(F)$ is small and assume instead that the singular values of F decay rapidly, so that $\operatorname{rank}_\epsilon(F)$ is small. Such scenarios occur in numerical computing [12–14], where (1) arises from the discretization of certain PDEs and F is associated with a smooth 2D function (see Section 6). Current methods only bound singular values of X with indices that are multiples of ρ (see Section 2),