Linear Algebra and its Applications 548 (2018) 66–76

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# A family of graphs that are determined by their normalized Laplacian spectra ☆

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A R T I C L E I N F O

*Article history:*
Received 1 November 2017
Accepted 1 March 2018
Available online xxxx
Submitted by R. Brualdi

*MSC:*
05C50
05C35

*Keywords:*
Normalized Laplacian spectrum
Generalized friendship graph
Spectral characterization
Cospectral graph

A B S T R A C T

Let $F_{p,q}$ be the generalized friendship graph $K_1 \bigvee (pK_q)$ on $pq+1$ vertices obtained by joining a vertex to all vertices of $p$ disjoint copies of the complete graph $K_q$ on $q$ vertices. In this paper, we prove that $F_{p,q}$ is determined by its normalized Laplacian spectrum if and only if $q \geq 2$, or $q = 1$ and $p \leq 2$.


☆ This work is supported by the Joint NSFC-ISF Research Program (jointly funded by the National Natural Science Foundation of China and the Israel Science Foundation (Nos. 11561141001, 2219/15), the National Natural Science Foundation of China (No. 11531001).

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https://doi.org/10.1016/j.laa.2018.03.001
0024-3795/

## 1. Introduction

Spectral graph theory studies the relations between the structure of a graph and eigenvalues of matrices associated with it. One of the main problems in spectral graph theory is which graphs are determined by their spectrum or equivalently, finding nonisomorphic graphs $G$ and $H$ that have the same spectrum. Many results on these questions can be found in two excellent surveys (see [9,10]) by Van Dam and Haemers.

Let $G$ be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_v$ be the degree of a vertex $v \in V(G)$. The *normalized Laplacian matrix* of a graph $G$ is defined to be $\mathcal{L}(G) = (l_{uv})$, where

$$l_{uv} = \begin{cases} 1, & \text{if } u = v \text{ and } u \text{ is not an isolated vertex;} \\ -\frac{1}{\sqrt{d_u d_v}}, & \text{if } u \text{ is adjacent to } v; \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $\mathcal{L}(\mathcal{G})$ are called $\mathcal{L}-$eigenvalues. This paper deals with the normalized Laplacian matrix of $G$, so we denote its spectrum of $G$ (the all eigenvalues $\mathcal{L}(G)$ of $G$, including multiplicities) by $Sp(G)$. We say that $G$ and $H$ are *cospectral* if they are not isomorphic, but $Sp(G) = Sp(H)$, and that $G$ is determined by its normalized Laplacian spectrum if $Sp(H) = Sp(G)$ only when $H$ is isomorphic to $G$.

Chung in [4] showed how the normalized Laplacian spectrum reveals fundamental properties and structure of a graph. Butler [1] surveyed algebraic aspects of $\mathcal{L}(G)$ and provided (see [2] and [3]) several methods of constructing cospectral graphs. Almost all small graphs are determined by their normalized Laplacian spectrum (see [3]). The normalized Laplacian spectrum of a complete graph $K_n$ is $0, \frac{n}{n-1}$ with multiplicities 1 and $n-1$ respectively, and $K_n$ is determined by this spectrum [3].

Butler [1] conjectured that the only graphs cospectral with a cycle are $K_{1,3}$ and the graph $\gamma_{4k}$ obtained by identifying the center vertex of a path on $2k+1$ vertices and a vertex of a cycle on $2k$ vertices; i.e., a cycle on $n$ vertices is determined by its normalized Laplacian spectrum if and only if $n > 4$ and $4 \nmid n$. In general, up to now, there are very few graphs that are known to be determined by their normalized Laplacian spectrum. In this paper, we present a family of graphs that are determined by their spectrum.

Denote by $F_{p,q}$ the graph $K_1 \bigvee (pK_q)$ on $pq+1$ vertices obtained by joining a vertex to all vertices of $p$ disjoint copies of the complete graph $K_q$ on $q$ vertices. The *friendship graph* $F_k$ consists of $k$ edge-disjoints triangles that meet in one vertex (for the famous friendship theorem, see [6]). Liu et al. [8] proved the $F_k$ is determined by its Laplacian spectrum (see also [7]). Wang et al. [12] proved that $F_k$ is determined by its signless Laplacian spectrum. Recently, Cioabă et al. [5] proved that $F_k$ is determined by its adjacent spectrum if and only if $k \neq 16$, and that for $F_{16}$, there is only one graph nonisomorphic to $F_{16}$, but with the same adjacency spectrum. The friendship graphs are a subfamily of $F_{p,q}$, since $F_k = F_{k,2}$.