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Proof of conjecture involving algebraic connectivity and average degree of graphs

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A R T I C L E I N F O

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A B S T R A C T

Let G be a simple connected graph of order n with m edges. Denote by $D(G)$ the diagonal matrix of its vertex degrees and by $A(G)$ its adjacency matrix. Then the Laplacian matrix of graph G is $L(G) = D(G) - A(G)$. Among all eigenvalues of the Laplacian matrix $L(G)$ of graph G, the most studied is the second smallest, called the algebraic connectivity $a(G)$ of a graph. Let $\overline{d}(G)$ and $\delta(G)$ be the average degree and the minimum degree of graph G, respectively. In this paper we characterize all graphs for which (i) $a(G) = 1$ with $\delta(G) \geq \lceil \frac{n-1}{2} \rceil$, and (ii) $a(G) = 2$ with $\delta(G) \geq \frac{n}{2}$. In [1], Aouchiche mentioned a conjecture involving the algebraic connectivity $a(G)$ and the average degree $\overline{d}(G)$ of graph G:

$$a(G) - \overline{d}(G) \geq 4 - n - \frac{4}{n}$$

with equality holding if and only if $\overline{G} \cong K_{1,\,n-2} \cup K_1$ ($K_{1,\,n-2}$ is a star of order $n-1$ and $\overline{G}$ is the complement of graph G). Here we prove this conjecture.

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1. Introduction

Throughout the paper we consider only simple graphs, herein called just graphs. Let $G = (V, E)$ be a graph on vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Also let d_i (or $d_G(v_i)$) be the degree of vertex v_i in G for $i = 1, 2, \dots, n$. The minimum vertex degree is denoted by $\delta = \delta(G)$ and the maximum by $\Delta = \Delta(G)$. Let D be the diameter of graph G. Also let N_i (or $N_G(v_i)$) be the neighbor set of the vertex $v_i \in V(G)$. We denote the complement of a graph G by $\overline{G}$. If vertices v_i and v_j are adjacent, we denote that by $v_iv_j \in E(G)$. For a subset W of $V(G)$, let $G - W$ be the subgraph of G obtained by deleting the vertices of W and the edges incident with them. The adjacency matrix $A(G)$ of G is defined by its entries $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $D(G)$ be the diagonal matrix of vertex degrees of graph G. Then the Laplacian matrix of G is $L(G) = D(G) - A(G)$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ denote the eigenvalues of $L(G)$. They are usually called the Laplacian eigenvalues of G. Among all eigenvalues of the Laplacian matrix of a graph, the most studied is the second smallest, called the algebraic connectivity of a graph. It is well known that a graph is connected if and only if $a(G) = \mu_{n-1}(G) > 0$ [11]. Besides the algebraic connectivity, $\mu_1(G)$ is the invariant that interested the graph theorists. Mathematical properties are studied on the largest Laplacian eigenvalue μ_1 and the second smallest Laplacian eigenvalue μ_{n-1} in [7,8,10,11] and several other papers.

In [1], Aouchiche mentioned a conjecture involving the algebraic connectivity $a(G)$ and the average degree $\overline{d}(G)$ of graph G:

Conjecture 1. [1] *Let G be a connected graph of order $n \geq 3$, average degree $\overline{d}(G)$ and algebraic connectivity $a(G)$. Then*

$$a(G) - \overline{d}(G) \geq 4 - n - \frac{4}{n}$$

with equality holding if and only if $\overline{G} \cong K_{1,\,n-2} \cup K_1$.

Several other conjectures on eigenvalues and graph parameters are available in [2–5].

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we characterize all graphs for which (i) $a(G) = 1$ with $\delta(G) \geq \lceil \frac{n-1}{2} \rceil$, and (ii) $a(G) = 2$ with $\delta(G) \geq \frac{n}{2}$. In Section 4, we prove Conjecture 1.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections. The following five results are on the upper bound for the Laplacian spectral radius of graph G.