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# A geometric mean for Toeplitz and Toeplitz-block block-Toeplitz matrices

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Using the symbol functions and their associated Fourier series, we introduce a new definition of geometric mean for all positive semi-definite Toeplitz matrices and positive semi-definite block-Toeplitz matrices with Toeplitz structured blocks (TBBT). The proposed definition preserves the structure of the matrices and satisfies some important Ando–Li–Mathias properties, such as monotonicity and continuity. Also, it has low cost and is simple to calculate.



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## 1. Introduction

Recently, the notion of geometric mean of scalars has been extended to positive definite matrices by many authors [1,5,6,8,14,16]. Since the straightforward generalization $(A_1 \dots A_n)^{\frac{1}{n}}$ does not satisfy many desired properties, even it is not invariant under permutation, new definitions of geometric mean for matrices have been developed, e.g. ALM, NBMP, CHEAP, and Karcher mean, see [1,4–6,8,16]. Because of the widespread applications of geometric mean in the many areas such as radar detection [15,21], image processing [20], elasticity tensor analysis [17] and medical imaging [11], many researchers involved in this field. Ando, Li, and Mathias [1], have suggested ten important properties, so-called *ALM properties*, that any geometric mean should satisfy them. But most of the definitions in the literature, do not satisfy all of the ALM properties, especially, they do not preserve the *monotonicity*. By the monotonicity property, we mean that $G(A_1, B_1) \leq G(A_2, B_2)$ whenever $A_1 \leq A_2$ and $B_1 \leq B_2$ where $A_1, A_2, B_1$ and $B_2$ are positive definite matrices and the operator $G$ stands for a given definition of 'geometric mean'.

Among the existing definitions, the Karcher mean satisfies all of the ALM properties but does not preserve the structure of the matrices, see [7]. More precisely, in many applications, such as designing of some radar systems, the used matrices are Toeplitz matrices [3]. A Toeplitz matrix is a matrix in which entries along their diagonals are constant, i.e.,

$$\begin{bmatrix} a_0 & a_1 & \cdots & & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & \\ \vdots & & \ddots & & \vdots \\ & & \ddots & & a_1 \\ a_{-n+1} & & \cdots & a_{-1} & a_0 \end{bmatrix}.$$

We are interested in finding a definition of a geometric mean of Toeplitz matrices that itself is Toeplitz too. Furthermore, it should satisfy some ALM properties, especially the monotonicity property. Using a Riemannian structure on the manifold of all positive Hermitian $n \times n$ matrices, D.A. Bini et al. [7] have introduced a (not necessarily unique) structured geometric mean which preserves the structure of Toeplitz matrices and satisfies a few ALM properties but fails to satisfy some other, especially the monotonicity property. Also, the Kähler metric mean [2,3] preserves the structure of Toeplitz matrices but does not satisfy the monotonicity property either. Moreover, the computations are somewhat complicated and costly.

The first author [19], introduced a geometric mean for positive semi-definite Toeplitz matrices with *non-negative* symbols which preserves the Toeplitz structure and many ALM properties, especially monotonicity in the sense of the order (3.1). More precisely, recall that every Toeplitz matrix has a unique *symbol* function, see [12] or section 2 below. Now, let $a$ and $b$ are the symbol functions of Toeplitz matrices $A$ and $B$ respectively.