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The Hurwitz-type theorem for the regular Coulomb wave function via Hankel determinants

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ABSTRACT

We derive a closed formula for the determinant of the Hankel matrix whose entries are given by sums of negative powers of the zeros of the regular Coulomb wave function. This new identity applied together with results of Grommer and Chebotarev allows us to prove a Hurwitz-type theorem about the zeros of the regular Coulomb wave function. As a particular case, we obtain a new proof of the classical Hurwitz's theorem from the theory of Bessel functions that is based on algebraic arguments. In addition, several Hankel determinants with entries given by the Rayleigh function and Bernoulli numbers are also evaluated.

1. Introduction

The Bessel functions as well as the Coulomb wave functions belong to the very classical special functions that appear frequently at various places in mathematics and physics.

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Since the regular Coulomb wave function represents a generalization of the Bessel function of the first kind, its properties are often reminiscent to the respective properties of the Bessel function. On the other hand, not all methods applicable to Bessel functions admit a straightforward generalization to the case of Coulomb wave functions.

A significant part of the research on Bessel functions and their generalizations is devoted to the study of their zeros. Naturally, methods of mathematical, in particular, complex analysis turn out to be the very well suited techniques in the analysis of the zeros that prove their usefulness many times during the last century. However, one of the main aim of this article is to stress the importance of linear algebra in the analysis of zeros of entire functions. By using primarily linear algebraic techniques, we prove a theorem on the reality and the exact number of possible complex zeros of the regular Coulomb wave function. The proof consists of two main ingredients. First, with the aid of certain properties of a particularly chosen family of orthogonal polynomials which is intimately related to the Coulomb wave function and was studied in [27], we evaluate the determinant of the Hankel matrix whose entries are given by sums of negative powers of the zeros. Second, we combine the formula for the Hankel determinant with general results of Grommer [8] and Chebotarev [29], which straightforwardly yields the desired goal.

The obtained result generalizes the well-known theorem of Hurwitz about the zeros of the Bessel function of the first kind; see Theorem 12 below for its formulation. Consequently, the present approach provides an alternative proof of this classical result, which is rather algebraic and simple. Let us remark that the original proof from Hurwitz [14], based on the connection between Lommel polynomials and Bessel functions, need not be easy to read nowadays since the modern terms were not introduced in his time. In addition, the original proof contained some gaps that were corrected later by Watson [30]. Nonetheless, the Hurwitz's result was very influential and many other proofs were found; see, for example, the articles of Hilb [10], Obreschkoff [23], Pólya [25], Hille and Szegő [11], Peyerimhoff [24], Runckel [26], and Ki and Kim [17].

Let us also point out that the methods developed in the papers cited above seem not to be readily applicable to the case of the Coulomb wave function treated here. A reason for this can be that, unlike Lommel polynomials, no explicit expression for the coefficients of the orthogonal polynomials associated with the Coulomb wave function is known. Rather than that, only a recurrence relation for these coefficients is available, see [27, Prop. 10].

The main results of this paper are Theorems 1 and 11, and the paper is organized as follows. In Section 2, we briefly recall necessary definitions and basic properties of the regular Coulomb wave function and an associated spectral zeta function which, in a special case, simplifies to the well-known Rayleigh function.

In Section 3, we prove a closed formula for the determinant of the Hankel matrix whose entries are given by the zeta function associated with the regular Coulomb wave function. As a corollary, we obtain formulas for determinants of two Hankel matrices with entries given by the Rayleigh function. These two Hankel determinants can be