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The weighted Kirchhoff index of a graph

Hideo Mitsuhashi [a], Hideaki Morita [b], Iwao Sato [c,*]

[a] *Department of Applied Informatics, Faculty of Science and Engineering, Hosei University, Koganei, Tokyo 184-8584, Japan*
[b] *Division of System Engineering for Mathematics, Muroran Institute of Technology, Muroran, Hokkaido 050-8585, Japan*
[c] *Oyama National College of Technology, Oyama, Tochigi 323-0806, Japan*

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ABSTRACT

We consider the weighted Kirchhoff index of a graph G, and present a generalization of Somodi's Theorem on one of the Kirchhoff index of a graph. Furthermore, we give an explicit formula for the weighted Kirchhoff index of a regular covering of G in terms of that of G.

1. Introduction

1.1. The Ihara zeta function, the complexity and the Kirchhoff index of a graph

Graphs and digraphs treated here are finite.

* Corresponding author.
E-mail address: isato@oyama-ct.ac.jp (I. Sato).

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Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. Furthermore, let $n = |V(G)|$ and $m = |E(G)|$ be the number of vertices and edges of G, respectively. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Let D_G the symmetric digraph corresponding to G. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$. For $v \in V(G)$, the *degree* $\deg_G v = \deg v = d_v$ of v is the number of vertices adjacent to v in G.

A *path* P *of length* n in G is a sequence $P = (e_1, \cdots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leq i \leq n-1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \cdots, n$, then we write $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$. Set $\mid P \mid = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$*-path*. We say that a path $P = (e_1, \cdots, e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e_{i+1}^{-1} = e_i$ for some $i(1 \leq i \leq n-1)$. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \cdots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \cdots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *Ihara(–Selberg) zeta function* of G is defined by

$$\mathbf{Z}(G, t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G. Ihara [12] defined Ihara zeta functions of graphs, and showed that the reciprocals of Ihara zeta functions of regular graphs are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [20,21]. Hashimoto [10] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [1] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Let G be a connected graph with n vertices $v_1, \cdots, v_n$ and m edges. Then the *adjacency matrix* $\mathbf{A}(G) = (a_{ij})$ is the square matrix such that a_{ij} is the number of edges joining v_i and v_j if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Let $\mathbf{D} = (d_{ij})$ be the diagonal matrix with $d_{ii} = deg_G\ v_i$, and $\mathbf{Q} = \mathbf{D} - \mathbf{I}$ two $2m \times 2m$ matrices. Furthermore,